INVESTMENT-CURVE MODEL OF TAX OPTIMIZATION AND TAX COMPETITION[1]

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Abstract

Theoretical framework for discussing mechanism of interregional tax competition in Russia is developed. Its models, designed both for analytical study and for empirical verification, exploit “investment curve” notion, that mean picture of immobile or imperfectly mobile investment projects, attached to each region. Regions choose their tax rates, behaving like price-takers in relation to country’s capital market. When they play only with general tax rate, one possible mechanism of *divergence* effect is due to inability of some regions to cover current needs if granting tax reductions. The effect can be absent, when regions discriminate between old and new capital, granting tax holidays. “Development *efficiency*” of tax competition is due to one-way direction of the described competition, that is tax reduction. Fiscal efficiency has the same nature as price discrimination benefits for monopolist, while welfare efficiency may be absent, for the same reason.

0. Introduction

We are interested in motives and mechanisms of interregional tax competition for investments in Russia, which is widespread now and expresses itself in various regional tax reductions, tax relieves and especially tax holidays. Main goal is to study divergence effect (asymmetry increase) and efficiency implications of the competition, that may give some hints for federalism policy.

For this goal we construct a specific family of appropriate variants of tax competition model[2]. All versions of our main model are in Tiebout tradition, being, specifically, close to Oates-Schwab model of *regional taxes on business* (see literature revue in the next subsection).

The difference of our approach from Oates-Schwab model is, that we consider incomplete mobility or even immobility of investment projects, described by "region's investment curve" notion, that is regional capital demand curve. We introduce also exogenous parameter of capital supply mobility (in contrast with Oates-Schwab constant

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[2] In essence, this theory is still an insufficiently organized collection of models, or model variants, supplemented by some links, by empirical and logical reasoning, it is not coming out of a single model. Rather, we have in mind something like hypermodel, connecting everything, but it is not easy to make it clear.

country's capital supply), and a parameter of “necessary" level of budget spending. This short-term “budget constraint" hampers possibility of some regions to grant reductions or tax relieves to obtain future capital and revenue gains.

These specific features are supposed to rely to Russian specific situation. Most real investment in current years of stagnation are domestic investments. Moreover, it is mainly implemented not by banks and other financial intermediaries, but by existing enterprises (only partially they find external financial support for their projects). It is due to very high transaction costs of financial markets: enterprises do not trust banks, banks do not trust enterprises, so profit goes either abroad, or into developing current business. As a result, there are more immobile investment projects among implemented, then mobile ones. Regional budgets’ tightness is well known also.

The interaction of regions is described as a dynamic game of 90 players (89 regions and capital market ("dummy" player, whose behavior is described by capital supply curve). Regions show Nash ("price-taker's") behavior in relation to country’s capital market, but behave like monopolists in relation to their local demand for capital. It is a close analogue of a game among many resellers of some commodity, buying it on the open market, but selling it monopolistically on their closed home markets. More specifically, in this repeated game all regions determine their taxes supposing current profit rate to persist, these taxes influence capital demand, then capital market determines temporary-equilibrium rate of profit, and so on. The equilibrium is defined as a bundle of tax rates and rate of profit, equalizing the supply of capital and cumulative demand of all regions, provided that it is not favorable to regions to change their current tax rates (Nash equilibrium among 90 players).

Rationality hypothesis: Conscientious, or, rather, "milking-and-hoarding" type[3] of rational behavior of regional authorities is supposed (alternative types, like "hit-and-run", or "behave-like-all" are neglected), that means pursuing tax revenue goals and development goals. More specifically, each region knows the set of its investment projects, i.e. its capital demand described by linearized "investment curve", and by capital-supply mobility parameter: alternative net-profit rate. So, regional capital market is almost like commodity market. Its Marshallian cross include some demand curve and constant marginal costs, while profit tax is like *ad valorem* tax, property tax is *unit tax*. The slight difference from commodity market is in tax-revenue formula: in contrast to *ad valorem* taxation, profit tax base exclude property tax revenue, that modifies somewhat the legislator’s behavior.

We suppose linear capital demand, for most results, or linearized demand (that is much more general and realistic), the difference is in some constants modifying profit tax revenue. Legislator optimizes either general rates of profit tax and property tax (simpler version, similar to choosing combination of *unit tax* and *ad valorem* tax on commodity market), or combination of these 2 tax rates and 2 corresponding tax relieves. Moreover, if we suppose legislator to be able to discriminate in taxing several (say, *n*) types or classes of capital within its region, then we should suppose not 4, but 4*n* optimization variables. However, there is no need to include this complication into dynamic model; it

[3] We mean that regional administration may be obsessed with tax revenue and development not only for conscientious reasons, but also for egoistic motives, trying to accumulate more resources in its discretion. Necessarily, this assumption, reminding “rational dictator” literature, implies long-term planning horizon of a dictator, otherwise “hit and run” behavior seems probable.

is almost[4] the same as just introduce 89n regions instead.

The objective function includes weighed sum of present and future tax revenue (cumulatively weighed), and present and future capital. The latter term describes the administration's goal of "hoarding", that is development or employment within the region, while the former displays "milking" goal. There is some tradeoff between the two. This problem, both in short-term and in long-term variants, is similar to "Laffer curve" problem, only the curve becomes a surface now, and two weighed goals instead of simple revenue are pursued. It is solved within legal constraints on tax variables, and within the budget constraint, determined by "vitally necessary" regional public spending.

Peculiarity of this tax-optimizing setting is motivated by our object: Russian capital. Nevertheless, similarity to other known optimization problems prevails, so some outcomes turn out to be rather typical, while some are problem-specific. Typical ones are similar to textbook monopolist's price-discriminating problem, and to Ramsey's tax-optimizing problem, they are: effectiveness of implementing tax discrimination and relative effectiveness of profit tax.

Tax discrimination within a Russian region is practiced in several aspects: among different industries (tax relieves for "weak" ones), between old capital and investment (tax holidays), between domestic and foreign investments (privileges for foreign capital) , and some other types. Theoretically, it makes sense taxing tax-inelastic groups (say, existing stable industries) more heavily, in comparison with tax-elastic ones, like new capital, depressive industries, or foreign capital. Indeed, Russian tax-discrimination practice clearly shows this tendency. It is no wonder, comparing our problem with standard price discrimination problem of a monopolist (Joan Robinson setting), or with similar Ramsey's optimal taxation setting (optimal combination of unit taxes for several commodity markets): both give the same advice.

How we should combine profit tax and property tax for some group of capital, say, small business, within which we can not further discriminate? Monopolistic theory have no advice for this case, while Ramsey's theory show preference for *ad valorem* taxation, since it implicitly discriminates more profitable subgroups within legally identical group. Our analysis do support this view, giving *preference to profit tax*, with some reservations for the case of "very high" sensitivity to taxes: the answer turns out to be not general, but value-dependent.[5] Besides, analysis shows some specific details of optimal tax combinations.

The motives of discrimination appear also in Section 2 considering dynamic game of inter-regional tax competition (but dealing only with comparative statics, see *general concept* in the beginning of the section).

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[4] The difference is in common regional-budget constraint connecting taxation problems of several groups of capital within region. However, we believe that it does not affect the results.

[5] Surely, here we ignore tax evasion considerations, favorable for property tax and for unit tax. In contrast with theoretical prescriptions to discriminate old capital that meet reality, the preference for profit-tax relief is not observed in Russia, may be due to fear of tax evasion.

There we assume no discrimination within regions (unless interpreting “regions” as 89n groups of capital, like above), but the result of tax competition is closely related to optimal decision of imaginary coalition of these regions, that chooses whole-country optimal tax bundle, and, naturally, discriminates.

Therefore Nash outcome turns out to be Pareto optimal for the 89 administrations, at least when short-term budget constraint is not binding. In essence, this fact is due to their price-taking position in relation to capital supply, as explained in more detail in informal proof of (hypothetical) Theorem 3, it is just 1-st Welfare Theorem consequence. This situation is quite opposite to the case with mobile investment projects and mobile capital, known as main version of Oates-Schwab model. If we assume in it (unlike Oates and Schwab) no business interest in local public goods, then we obtain antagonistic game among regions, resulting in famous “race for the bottom”, i.e. zero equilibrium taxes.[6] In our model we escape this disease due to immobility.

However, it does not mean that study of our model gives additional unambiguous theoretical arguments in favor of fiscal federalism, rather they are controversory. First of all, efficiency notions are multiple, secondly, different assumptions yield different outcomes, and thirdly – divergence effect may be supposed by some policy-makers to overweigh efficiency benefits. We express these results (some of them are supported only by informal reasoning or examples yet) as follows.

Study of game-theoretical tax competition model has shown, under the specified assumptions, that the competition has following consequences for efficiency and regional asymmetry. A stationary point of this economy (an equilibrium) with tax competition differs from an equilibrium without tax competition by the following features:

- When region’s short-term “budget constraint” does not play active role (that is the case, in particular, when tax holidays are practiced), *administrative efficiency* is achieved, i.e. goals of regional administrations achieved in equilibrium can not be Pareto-improved. This implies *fiscal efficiency*, i.e. maximal total country’s tax revenue – in special case when administrative goals consist only in long-term budget revenues, and when capital demands are linear. *Development efficiency* of competition in the latter case appears if and only if initial uniform tax rate was not optimal. *Welfare efficiency* (total consumer’s and producer’s surplus, plus tax revenues), on the contrary, may be enhanced or deteriorated by competition, depending upon initial state: if initial state was optimal and all markets functioned, then competition makes losses.
- Take the case when region’s short-term “budget constraint” does play active role only for some poor regions (suppose no tax holidays, demand linearity, and other restrictions), while tax competition goes one-way, starting from the highest admissible level of tax downwards. Then *development efficiency* of competition is present, together with *divergence* (increasing asymmetry), and with usual presence of *low-budget traps*, that is isolated low-capital equilibria for some or all regions, which can be avoided if having more initial capital.

[6] This tendency is observed indeed, but mainly in relation to some headquarters, moving to Altai and Ingushetia where taxes are very low, while we study real investment.

Russian situation seems more close to the first of the two mentioned above situations: regional tight budget constraints can be overcome by tax holidays. So, we should expect no divergence of this type and no low-budget traps. However, traps (they are most interesting for policy-makers, may be, among all effects) can well appear due to other realistic features, not included in this version (see Fig.3 with comments).

Presentation goes as follows. Small subsection 0.1. presents some review of literature and discussions around efficiency or inefficiency of fiscal federalism, and about realism of different type of models. It explains our choice of model, and helps to position of our study in this discussion: in essence, our paper enriches complex picture of various effects, born by different assumptions.

Section 1 consisting of several subsections concerns the basic element of tax competition models: region's rational behavior model. Subsection 1.1 explains the "investment curve" concept and related tax discrimination possibility. Subsection 1.1 introduces linearized investment curve and related formulae for finding investment volume and tax revenue depending upon taxes. Next subsection generalizes the approach to dynamic tax optimization formulation, and the last one describes solutions to the formulated problem.

Section 2 presents several dynamic tax competition model versions, starting with general construction and rough explanation of its performance. Then goes the simplest version of the model, that allows for analytical results including divergence theorem, and more elaborated models, with numerical examples and hints.

0.1. Review of literature: **models of interregional tax competition and discussions**

In federalism studies, there are different modifications of Tiebout's tax competition model (Tiebout, 1956). The basic model assumes mobile participants of economy (interpreted normally as households) who have different preferences concerning public goods: transport service, police, education and others. Regional authorities are supposed to be interested in attracting economic agents (tax-payers) by producing public goods with some costs covered by tax revenue. Then for "large" number of regions, their Nash competition for tax-payers must yield two effects: 1) Pareto-optimal level of taxes and public goods in every region; moreover the taxes prove to be analogy of market price of public good (benefit taxes), they can not be used for redistribution: transfers to poor, etc.; 2)divergence, i.e. heterogeneity of regions with respect to all parameters: each region is supposed to become a community ("Tiebout club") of tax-payers with approximately same interest in public goods. These ideas strongly support federalism.

This theoretical model, being appropriate mainly for USA, became a starting point for different modifications, also supporting tax federalism for efficiency reasons ("conservative" point of view).

In particular, analogous efficiency conclusions were obtained for the case of immobile population but capital mobility (Oates W., Schwab R., 1988). Controversial is theoretical model by J.D.Wilson (Wilson J.,1987) describing tax competition in the presence of two technologies: capital-intensive and labor-intensive. It shows that the competition induces divergence of regions into capital-intensive (applying low tax rates) and labor-intensive (applying high tax rates) Tiebout groups, that can be inefficient due to excessive transportation emerging and to wrong level of public good provision. However, the revealed inefficiency is connected first of all with firms being indifferent to public goods. This assumption would bore the same effect in Oates –Schwab model, moreover, then competition results in zero taxes, that is named “race for the bottom”.

Opponents of tax federalism ("liberals") criticize it mainly on egalitarian grounds, defending large budget of social transfers, which they propose to finance at the expense of taxes on business and on rich people, that is impossible in Tiebout game. Conservatives object that it is the federal budget destined for social transfers. Actually it is not the case: many regional authorities also practice essential redistribution, financing the poor at the expenses of the reach (Reschovsky A., 1991). Tiebout forces should punish this practice. Indeed, some empirical investigations confirm the hypothesis about firm's flight from regions using redistributional programs (not simply high taxes, but taxes for redistribution (Oates W., Schwab R., 1988)). How should we interpret these facts? Regional practice of redistribution, contradicting to Tiebout's scheme of competition, may be evidence of too slow work of Tiebout forces, or evidence of insufficient adequacy of Tiebout model even for USA.

There are several factors, whose introduction into tax competition model could explain the named gap with reality, among them imperfect taxpayer’s mobility (high costs of reallocating existing firms, tec.).

Further, there is also important influence of *political* motivation of regional authorities providing deviation from pure Tiebout's competition. In particular, in (Shannon J, 1991) the effect of “pacesetter phenomenon" is described, also called "Stockholm syndrome": usually “a piece of radical legislation gets passed by the Swedes, then it's flown directly to the U.S. and is passed into law in California. Then it's flown to Wisconsin. Then to New York. By the time it gets to Mississippi, which is about four years later, it's a national birth-right." This political effect is very prominent in financing educational programs (which provide not only redistribution but also benefits for firms and for households). Those states, where historically level of expenditures on education is not high, are not satisfied with (prescribed by Tiebout’s model) role of just being a “club" for firms and households not demanding education. These states prefer to *imitate* their educated neighbors and make super-efforts not to fell behind in this field. As a result of the imitation, taxes and expenditures on public goods increased gradually (very significantly) in 20-th century in *all* states. Now states look like one compact cluster in this field (“a convoy"), in contradiction with Tiebout prediction.

The point is that politicians are afraid of *political critique*, which is therefore a factor mitigating Tiebout tax competition effects. In essence, not only consumers has influence on regional authorities, “voting by legs" according to Tiebout model, but also electorate voting with bulletins. Elections counterbalance anti-redistributional Tiebout-like stimulus of regional authorities. Exactly this counteraction of the two stimulus is approved by conservative theorists, defending tax competition: it does not allow voting poor to rob too much the rich, and, what is more important for us, to rob business (affecting the future population of this region). These reasons become especially important in Russian conditions because of prevalence of poor politically shortsighted voter.

Observations of "imitation" practice make possible to agree with applicability of "diffusion of innovations " models to the considered area, proposed in the studies (Coleman J., Menzel H., Katz E., 1959), (Reinganum J., 1981), (Salmon P., 1987). The basic assumption, in contradistinction to Tiebout model is incomplete information/rationality of the participants and other factors of delay ("stickiness") in the innovations spreading. It is recognized, that they produce S -shaped or logistical curve of innovations spreading in time: rare first examples, then mass application, fading on rare exclusions. Besides incomplete information, a factor of delay is slowness of political process in each region. These considerations make us doubt about what we observe in each case: an equilibrium, or just slowly moving disequilibrium. Changes in incomes, in demand and in technologies in the post-war period goes by rates comparable with the rates of tax innovations, that does not allow to see a stable state. It is considered to be an observable fact, and it is incorrectly named “equilibrium instability" in (Breton A., 1991), mixed with race for zero level of the taxes - "race for the bottom" (in particular, the reduction of income taxes in megapolice suburbs is observed).

Another deviation from Tiebout concept of regional authorities’ motivation is described as “cooperative federalism” (Elazar D., 1991). Like competing oligopolists, regions are capable in some moment to switch onto cooperative behavior. Difference is that such behavior of regions is ideologically approved, it can be maintained by federal authority and can have legislative mechanisms for the agreements maintenance. Cooperative behavior is observed in many inter- regional joint projects and affairs. Nevertheless, in the field of tax competition it is supposed to appear only when mutual losses from competition will not be considered by the participants as “rather significant ". This *threshold* is supposed being low in Canada (Elazar D., p. 66), with its few regions, where it is easier to reach an agreement. Potential agreement, which can be registered as federal legislative act (without the consent of part of regions either), is a discreet political event, interrupting, not smoothing the interregional competition. So, it can be incorporated into tax competition model only as a threshold, stopping it.

Among the studies of Russian regional competition for business, close to our topic is the stochastic model of tax-holidays optimization by a single region (Arkin, V., Slastnikov, A., Shevtsova, E., 1999) (Аркин В., Сластников А., Шевцова Э., 1999). In contrast with our study, optimal choice is not coordinated with budget needs and two-tax mixture, focus is on stochastics.

Vizhina's (Vizhina I., 1998) (Вижина И., 1998) paper analyzes laws, resolutions and decrees. Interesting paper of L.Polishchuk (Polishchuk L., 1999) reveals motives of tax relives, and doubt, by the way, in their efficiency. Paper (Kuznestova, 1998) comparatively study two similar neighboring regions: Novgorod and Pskov. It states important role of regional tax policy, and drastic superiority of tax-holiday granting region – Novgorod, expressed in all economic indices.

Relying on different ideas mentioned, we stopped our choice of tax competition model for Russia on some modification of Oates-Schwab model, characterized by imperfectly mobile capital.

1. Tax optimization on investment curve: use of discrimination and two-tax combinations

We start our study of tax competition with revealing the structure of tax optimization problem solved by a *single* regional authority, referred further as Legislator. Then this rational behavior model becomes an element of the game model.

1.0. *Clarifying remarks*

We shall argue only in gross-regional-product terms. Thus we abstract from essential but complicated inter-industry differences. We shall speak mostly of "investment", but one can easily transfer the same argument to the problem of underloading the existing industrial capacities, if they are sensitive to taxes, and, with some reservations, to taxing commodity markets. Simplifying further, we shall optimize w.r.t. only 2 tax variables: property tax and profit tax. Other taxes, including most important VAT, are supposed to reducible to combination of the two taxes considered (see Appendix) .

Many realistic considerations, that can well abolish applicability of our deduction, like corruption, irrationality, etc. – are beyond the scope of this study. We do also abstract from risks and from time schedule of gathering taxes, all these are supposed to be reduced on *mean-present-value* basis to comparable permanent-profit terms. Thus we ignore the fact that tax holidays are usually preferred by entrepreneurs to present-value-equivalent permanent tax relieves, we display both in the same manner as "tax relieves". Accordingly, "net profit" notion below will mean "steady warranted profit" (depreciation subtracted), comparable to the interest rate that an investor may get for reliable tax-cleared securities in US or Europe, say 5-6%.

Further, we shall consider here only conscientious-rational behavior, that is maximizing development and tax revenues of the region, we do not touch other types existing.[7] Therefore, the sense of our study is to reveal effects connected with this type of behavior only, putting aside other effects.

1.1. *Basic investment curve and mobility curve, "perfect" discrimination*

[7] Relying on L.Polishchuk's paper we can construct at least four hypothetical models of legislative and administrative behavior of regional authorities, apparent plausible. We shall describe them in stylized manner. 1) "Novgorod" model (long-term altruistic, rational): the administration have professional motives, aspiring future well-being of the region, it designs consecutive strategy of attracting investments. 2) "Ulyanovsk" (short-term rational) model: the administration tries to achieve current well-being, together with success in the nearest elections, it deserves preservation of employment and guarantees to the population at any cost, including significant sacrifices from the profitable enterprises, subsidies for unprofitable ones. 3) "Pskov" (predatory) model: the administration, aspiring to increase its personal authority and property, regularly demonstrates to the enterprises its own ability them to rise or ruin them, to grab share of assets - or just political loyalty and support. 4) Irrationally-imitative model: regions just copy patterns of behavior from their neighbors, patterns which are considered "progressive" for some reasons. This can be born by information imperfections of the political game. At the given stage we could only empirically confirm that behavior of 1-th and 4-th types is present. We also study some variants of the 1-st type analytically.

We need formula for expected investments in a region, depending upon profit and taxes. It should describe regional *demand* for capital, that is investment possibilities. Let us construct this "investment curve".

Suppose, there are some investment projects, characterized by investment volume and gross profit. After ordering them in descending manner, we get the needed function *I*, where *I*(*p*) describes total possible investments with profit exceeding *p*. For instance, such function born by 3 investment projects, having volumes 30, 15, 20 - see on Fig.1. (for realistic samples see Appendix).

Meeting this demand-curve with a capital-supply curve, presented on Fig.1 by straight line, should give equilibrium, i.e. actual investments. Such flat supply -curve is defined by fixed "alternative-profit' rate", i.e. the net profit rate available for investors somewhere else. In some cases this line may be not straight, due to different "alternative-profit' rates" for different projects.

How much tax revenue regional administration can earn facing given investment curve and supply-curve? It depends upon its ability to discriminate among projects, and upon alternative profit that each project can earn in another region. Supposing that alternative profit of all the investors in question is the same (vertical line on Fig.1), and that the administration has legal and intellectual ability to fully discriminate, then legislator can grab the whole potential profit of the region without harm to investment and to production!

On Fig.1 this maximal-possible tax revenue becomes the whole integral between the investment curve and the line of alternative profit. Then the taxes levied upon all projects (property taxes, on this picture) are equal to expected profits. This solution is a direct analogue of well-known "ideal price discriminating" by monopolist. Some its version, though restricted by federal law, is practiced in Novgorod region (see Appendix).

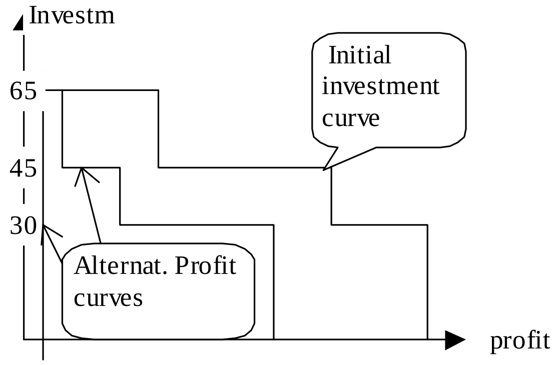

Figure 1. Investment curve and two different alternative-profit curves: with uniform mobility and non-uniform mobility. Maximal possible tax revenue is the integral between the investment curve, and the alternative-profit curve**.**

Possible non-uniform alternative profits of the projects describe their *mobility*: when projects' alternative profit is uniform, it means uniform mobility. To calculate maximal-possible tax revenue in uniform or non-uniform case we should just subtract mobility graph from capital demand graph, and take the integral between the two curves. Thus, generally, capital demand curve and mobility allow us to see full taxable resources of a region.

The monopolistic arguments show, that legislator in this position *can overcome Lafferian contradiction* between tax revenue and production volume, if being able to discriminate among projects (units of capital); discrimination enables to combine both goals on maximal level. This practice has some support among policy-makers, even for being implemented in the whole country. In essence it is a rent-extracting practice. Its most noticeable example is production-sharing agreements with oil-, gas- and other extracting enterprises. Observing obvious fiscal arguments in favor of tax-discrimination practice, we see only minor welfare arguments against it by now. The only clear welfare-based counter-argument is that such practice, equalizing net profits, can slow down the effective flows of capital from less profitable to more profitable industries and sectors, but this effect appears only at the limit, for complete discrimination, that is not practiced.

Even most smart and organized regional administrations usually manage to discriminate in taxation only among following groups of production: 1)between existing facilities and new investment (most important in Russian practice division); 2)among individual large investment projects, by fixing different ``repay dates" for them (``Novgorod model", see Appendix); 3)between large and ``small" business; 4)among several industries, in favour of depressive ones, while region-large enterprises are sometimes treated similarly (by special legislative act); 5)among several localities, in favour of depressive ones; 6)between domestic and foreign capital (taking into account different mobility of them).[8]

By these simple considerations we conclude discussing the discrimination topic, well known to theorists in other context. In closest paragraphs we shall not mention discrimination at all, except the discrimination between existing facilities and new investment. The point is, that tax-optimization problem can be (if putting aside unique “budget constraint" of our region) decomposed into several optimization problems: one task for each class of enterprises, each with its own “investment curve". If further discrimination within the given class is supposed impossible, then that constitutes the non-trivial problem of optimal tax mixture to be solved now.

As well as discrimination, we shall not further consider non-uniform mobility curves in this study.

1.2. *Linearized investment curve.*

Consider capital-demand curve of a single class of enterprises (or of the whole region, if we are not able to discriminate). In the Appendix examples the class is “new capital”, i.e. investment. Let us assume that the piece of the curve, supposed being affected by possible regional tax shifts - is not too tricky, that its concavity or convexity does not play essential role. Then we can linearize the curve on this interval as described

[8] Once again, we warn here not to mix the discussed here logical outcomes of theoretically-possible honest regional behavior with Russian present corruption in discriminating capital.

in the Appendix. However, we do linearize the curves even in spite of their tricky character. After all, it is only general expected *investment sensitivity to taxes* that matters for regional administration. It is a stochastic notion. Kinks and jumps of investment curves may fluctuate from one year to another, while the taxation must be stable, reflecting average tendency, so the average curve's slope matters more then temporary kinks.

Let us explain the notation related, and the formulae born by this triangular capital-demand model. A linear investment curve is determined by two constants: 1)maximal possible investment[9] denoted I^m *or* I^{max} ; 2) normalized "slope constant" g, such that expected investment is $I(p) = I^m \cdot (1\text{-}gP)$, when the required by investors alternative profit rate P is given. Without scale factor I^m the investment curve (1-gP) will be called normalized one, it relates to technology of the region, while scale factor – to size of the region. Maximal possible gross profit is P^m=1/g (that is the profit of the best available project on the triangle).

It means, that the more profit rate investors desire – the less investment they will make in our region, according to available regional projects. In more correct terms, this curve describes *gross* profit rate $P(I) = P^m\text{-}I/g$ that I-th small unit of capital can yield, being implemented in our region. In contrast, investors are comparing regions by alternative *net* profit. Let us express its impact on investment.

Suppose absence of taxes and an alternative net profit (profit available in some other region for investors) being π . Then, surplus rate of profit S (in comparison with an alternative region) that the I-th unit of capital can yield in our region is $S(I) = (P^m\text{-}I/g) - \pi$, so, equalizing $S(I)$ to zero, expected normalized investments are $I = (1/g - \pi)g$.

Now recall that our linear investment curve (triangle) may be born by linearizing more complex curve on the interval affected by taxes. Then a compensating constant P^c (positive or negative) must be added to gross profit P^m=1/g of the best triangle's project, so that real $P^{max} = P^m + P^c$. Accordingly, we should add some compensating constant C^I describing the integral between the real and the linearized curve (see Appendix), when calculating the taxation base for profit tax (however, we shall suppose C^I=0= P^c in special cases).

1.3. *Formulae describing impact of two taxes and tax revenue*.

Now, if we introduce property tax denoted x<1, the profit surplus of I-th project discussed above becomes $S(I) = (P^m\text{-}I/g) - \pi - x$. Further, when profit tax y is also present, the same surplus takes the shape $S(I) = (P^m\text{-}I/g) - \pi - x - (P^m\text{-}I/g\text{-}x)y$. We have taken into account, that taxation base for profit tax is not the gross profit of I-th capital unit itself (P^m-I/g), but the reduced volume (P^m-I/g-x), i.e. the volume corrected by property tax.

Accordingly, having in mind the supply of capital presented by alternative net-profit rate π, we can derive the number I of the last (worst) unit of capital implemented in our region. We should solve w.r.t. I the following linear equation, describing investor's choice:

[9] Notation: Upper index-letters will be used throughout as accents everywhere, not as power-marks or indices, unless being numerical or parenthesized. All indices will be lower ones, powers – in parentheses.

$$S(I) = (P^m\text{-}I/g) - \pi - x - (P^m\text{-}I/g\text{-}x)y=0 \ .$$

Solution for normalized investment is

$I = \boldsymbol{I}(x,y) = g(P^m - \pi - x - yP^m + yx)/(1-y) = (1 - g\pi - gx - y + gyx)/(1-y)$,

so real investment is[10]

$I^m\, \boldsymbol{I}(x,y) = I^m(1 - g\pi - gx - y + gyx)/(1-y)$.

Further, when total volume of investment implemented in our region is a number $I= \boldsymbol{I}(x,y)$, then the expected normalized annual revenue from property tax levied on all these investments is $x*\boldsymbol{I}(x,y)$. Let us derive region's normalized revenue from profit tax analogously. We see that for linear investment curve the best available profit-tax base, that is the profit rate of the best unit of capital, or of the best investment project, whose number is 0, is $P^m\text{-}x=1/g\text{-}x$ (property tax subtracted).

Naturally, the worst profit-tax base is the profit rate of the worst project numbered $\boldsymbol{I}(x,y)$, that is $\boldsymbol{I}(x,y)/g\text{-}x$ (property tax subtracted). Then, exploiting linearity, we get the average profit-tax base magnitude $B(x,y) = 0.5* \boldsymbol{I}(x,y)(1/g\text{-}x+ \boldsymbol{I}(x,y)/g\text{-}x) = \boldsymbol{I}(x,y)(1/2+ \boldsymbol{I}(x,y)/2\text{-}xg)/g$. However, if non-linear was the investment curve before our linearization, then that real maximal profit differs from our parameter P^m, and some constant C^I should be added to tax base, so, generally the normalized tax base is $B(x,y) = \boldsymbol{I}(x,y)(1/2+ \boldsymbol{I}(x,y)/2\text{-}xg)/g+C^I$.

So the normalized profit-tax revenue of our region is $(y\text{-}Y^{fed})(0.5\cdot \boldsymbol{I}(x,y)(1\text{-}\boldsymbol{I}(x,y)+2xg)/g+C^I)$ (we subtract here quantitity Y^{fed} =0.13 i.e. profit tax belonging by law to federal budget). Then total normalized tax revenue from investments R^I, related to property tax and profit tax is

$R^I=\boldsymbol{R}^I(x,y) =x\cdot I+(y\text{-}0.13)\cdot (I(1/2+I/2\text{-}xg)/g+C^I) =$

$=I\cdot [x+(y\text{-}0.13)\cdot ((1/2+I/2\text{-}xg)/g)]+(y\text{-}0.13)C^I$,

where investment volume I is derived as $I= \boldsymbol{I}(x,y) = (1-g\pi - gx - y + gyx)/(1-y)$.

Finally, we substitute now I, and change also specific values of federal share in property tax X^{fed}=0, and in profit tax Y^{fed}=0.13 for their general form. Then we obtain general (applicable for different countries) tax-revenue-from-investments formula in terms of initial parameters (normalized):

$\boldsymbol{R}^I(x,y) = (1-g\pi - gx - y + gyx)/(1-y)[(x\text{-} X^{fed})+(y\text{-} Y^{fed})\cdot ((1/2+$

$+((1-g\pi - gx - y + gyx)/(1-y))/2\text{-}xg)/g)]+(y\text{-} Y^{fed})C^I$,

while the real total revenue is $I^m\boldsymbol{R}^I(x,y)$.

Exactly the same logic as with investment, we can apply to find volume of operating *old weak* facilities (capital) K^w and, accordingly, to find revenue from this capital. Saying “weak”, we mean industries and firms being close to bankruptcy, we oppose them to “strong” or “normal” industries, touched later. All existing weak facilities we denote K^w,

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[10] Consider alternative case: different taxation law, when taxation base for profit tax does not exclude property tax. One can check, that similar investment formula would look like $I^m\, \boldsymbol{I}(x,y) = I^m(1 - g\pi - gx - y)/(1-y)$. This case is equivalent to taxing commodity market with mixture of *unit tax* (analogue of profit tax) and *ad valorem tax* (analogue of profit tax). So, it is (+gyx) component, the minor difference in the two formulae, that prevents expanding our analysis onto commodity markets.

that is the scale factor similar to former parameter I^m. "Weak" facilities may be under-loaded or can be closed. So, production *is sensitive* to taxes (x,y) with some sensitivity factor q, and sensitive to alternative closure-profit p^K that the owner can earn closing and selling these capital units (often below we suppose $p^K=0$). Then, we derive normalized weak capital as:

$\mathbf{K}^w(x,y) = (1\text{-}qp^K\text{-}qx\text{-}y+gyx)/(1\text{-}y)$, and scaled weak capital is $K^w\ \mathbf{K}^w(x,y)$, formula for tax revenue $\boldsymbol{R}^{Kw}(x,y)$ gained from weak capital is similar to $\boldsymbol{R}^I(x,y)$, (please, note, that our normalized *functions* ***I*,*R*,K** are bold-faced, while numbers are in normal font).

Now we should explain similar formula for tax revenue from "strong" or "normal" old capital. The formula must be based on trapeze, rather then on triangle, in contrast with previous ones. Indeed, postpone issues of complex dynamics, and suppose, that our region have long been in stationary position: the same alternative profit rate π, the same taxes (x,y), the same investment possibilities (I^m, g). Then, if not taking into account other circumstances, our old-capital profile must have exactly the same shape as our investment triangle, only without the upper part, i.e. without the projects, whose gross profit is less then $(\pi+x)$. It is a trapeze. Another difference from the investment triangle is that scaling factor for normal old capital must be $1/(1\text{-}a)$ times more, if $0<a<1$ is the depreciation factor. That is $K^s= I^m /(1\text{-}a)$, say, capital is10 times more then investments annually.

This *flat* roof of capital demand curve is the cause of important for our study effect: normal old capital must be *insensitive to taxes*, (at least in some vicinity of previous taxes)! Indeed, suppose that all investments for the last 10 years were accomplished expecting 5% normal net profit and 2% usual property tax. Assume no unexpected events in business for 10 years (not realistic assumption, of course). Then there is no capital in our region with gross profit less then 7%. So, decreasing taxes have no impact on currently operating capital at all, and increasing property tax have no impact also, until it exceed 7%, that is until it completely rob out normal profit! We shall assume such excesses not probable, so operating normal capital will be just a *constant* equal to all previously accumulated normal capital K^s.

Accordingly, normalized tax revenue from it will be (we calculate trapeze square, diminished according to "reduced tax base" rule, to find revenue from profit tax, the trapeze being the part of investment triangle):

$\boldsymbol{R}^{Ks}(x,y) = (x\text{-}\ X^{fed}) +(y\text{-}Y^{fed})\ 0.5\ (\pi+1/g\ \text{-}x)(1\text{-}g\ \pi\text{-}g\ x) + (y\text{-}\ Y^{fed})C^I$, while real revenue is $K^s\ \boldsymbol{R}^{Ks}(x,y)$.

The model would be more accurate, if we used not current net profit rate π in this formula, but different profit for each portion of old capital, namely the rate that was operating, when this portion was born, tracking back all the history. However, we believe that normal profit rate in the country do not change too essentially, so such complication of the model looks excessive.

Now, where "weak" capital comes from? The model can not do without additional assumption, in particular, we suppose exogenous parameter of "dispersion" $0<d<1$ – this is the share of normal capital that stochastically annually becomes "weak" due to different market circumstances: $K^w = d\ K^s$. This assumption we throughout.

In this case, sensitivity of total (weak and strong) old capital to taxes depends upon two parameters: d,q. When d is small, sensitivity is small, but generally it can be larger or smaller then sensitivity g of new capital. The sensitivity difference is crucial for choosing between tax holidays, and general tax reduction: when g sensitivity is larger, then there is more need in tax holidays.

We resume, total current real tax revenue is ($K^w\, \boldsymbol{R}^{Kw}(x,y)+ K^s\, \boldsymbol{R}^{Ks}(x,y)$). It should not be added to revenue from investments $I^m\boldsymbol{R}^I(x,y)$, because the latter describes *future* revenue.

Now with the help of these formulae we should argue about optimization patterns.

1.4. *STATIC OPTIMAL* **2**-*TAX MIXTURE*: *tradeoff between employment and tax revenue*

Let us abstract for a while from tradeoff between current and future objectives, from all other time considerations. Suppose, our region choose "only once, today" its taxation strategy (x,y), concerning only investments of *one* future period, influenced by taxation. This period, when the taxation will operate, can be next year after the decision, or another, the decision does not influence present time.

Our region may be supposed to have development or production objective (investment as such), revenue objective, and employment objective. However, if we suppose employment to be connected with capital (investment) by a simple linear relationship (production yield employment), then there is no tradeoff between these two objectives, so no need in introducing separate employment variable. In contrast, there can be tradeoff between current tax revenue objective and production/employment objective. This tradeoff is well known as the upward wing of the Laffer curve. Therefore, a region should compare (weigh) somehow these two objectives in its utility function. We suppose the simplest linear form, with weight denoted $w_3 = w^r \in [0,1]$ for tax revenues, and with weigh $w_4 = w^e = (1-w^r)$ for capital or investment as such (i.e. for employment, or production). Then objective function takes the form

$$U(x,y) = w^r\boldsymbol{R}^1(x,y)+(1-w^r)\boldsymbol{I}(x,y),$$

where functions $\boldsymbol{R}^1, \boldsymbol{I}$ are derived above. This optimization problem can be viewed upon as a version of Laffer problem, generalized in respect that now we face a Laffer surface $U(x,y)$ instead of a Laffer curve $U(y)$, and that we combine 2 objectives instead of one revenue objective.

We should optimize this function w.r.t. tax variables (x,y), subject to legal and natural constraints:

$$X^{min} \le x \le X^{max}\ ,\ \ Y^{min} \le y \le Y^{max}\ .$$

Initially we supposed reasonable to take only legal bounds on property and profit taxes $0=X^{min}$, $X^{max}=0.02$, $0.13=Y^{min}$, $Y^{max}=0.35$, prescribed by Russian law. Later we generalized the setting, and included also the parameter $X^{fed}>0$, to describe VAT component going to federal budget (see explanation in Appendix). This modification changed the bounds, but not the essence of constrained maximization[11].

One additional constraint is "budget" constraint:

$$G \le \boldsymbol{R}^1(x,y),$$

[11] In principle, the lower bounds on taxes can be overcome by subsidies, but it is not practiced. Exception is Novgorod'd region promise to subsidise federal taxes to new enterprises in depressive subregions, but none invested yet.

where G denotes necessary regional budget spending in the considered year. Somebody would object that most regions in Russia do not cover their expenditures. Then G_r should denote the volume that really "must" be covered, in opinion of the authorities (after taking into account transfers from federal budget and available debts and arrears).

The constraint should also participate in optimization, but it can be reflected simply by including a Langrange multiplier ρ , just adding it to the revenue weight w^r. The tighter is the budget constraint, the greater is ρ , and, accordingly, the weight of revenue objective. Until turning to country's dynamics, it makes no difference for us, how $(\rho + w^r)$ changes, so we can simply suppose ρ included into w^r during studying the Lagrangian, and impose no explicit budget constraint.

Therefore, the corresponding Lagrangian (with dual variables l_i , m_i) becomes

$$L(x,y, l_1, l_2, m_3, m_4) = U(x,y) + l_1(x - X^{min}) - l_2(x - X^{max}) + m_3 (y - Y^{min}) - m_4(y - Y^{max}).$$

In the next subsection we shall formulate more general problem, including this one as a special case, then discuss the solutions.

1.5. *DYNAMIC TAX-OPTIMIZING SETTING*: *tradeoff between current and future goals*

Now let us take into account dynamic aspects. It deserves optimizing taxes on new capital (investment) and on old capital simultaneously. This can be done in several reasonable fashions. We shall try to explain why some simplified solutions seem plausible, and how we should convert dynamic setting into the pseudo-static one.

1.5.1. *Stable-expectations stable-strategy setting* (*but for changing capital*)

We imagine a farsighted administration, trying to chose taxation scheme today for the whole future, having in mind the plan period T (may be, $T = \infty$). We does not mean that if conditions tomorrow change, the legislator will not change the solution. He/she probably will, but it will be again a solution for the whole new horizon T , expecting stable conditions again.

We may speak here of optimizing taxes for the whole region, or for a single industry, logic is the same. It can be also a special class of business, chosen to be taxed; let us speak of an ``industry". Suppose that for the industry our legislator know its current capital stock of old capital K^w ,K^s (weak and strong)**,** and all other relevant parameters q, C^K, p^K. In addition, let legislator expect stable future investment curve parameters I^m, g, C^K, π[12], and a stable capital depreciation parameter $a<1$.

Generally speaking, having in mind the plan period T, now very many (namely, $4*T$) taxation variables should be chosen simultaneously instead of former 2 variables. Indeed, for each year t we can chose property and profit taxes (x^{tK}, y^{tK}) for existing facilities, and also taxes (x^{tI}, y^{tI}) for new capital (investment), since our region can discriminate between old and new capital.

[12] Notation $p = p^K$ everywhere mean profit, like π. Stable parameters mean, that new emerging investment projects constitute each period the same new investment curve, projects are not exhausted.

However, it is too complex a problem, both for us and for legislators. Therefore, we shall neglect the possibility[13] to chose taxation strategy as a non-trivial trajectory $(x^{tK}, y^{tK}, x^{tI}, y^{tI})_{t=1}{}^{T}$. We believe, that legislation can not be changed each year. Hence, we shall optimize "once and for all times", so we need only 4 variables (x^K, y^K, x^I, y^I). May be, they should obey the constraint $(x^I, y^I) \le (x^K, y^K)$, if the law permits only tax holidays for investment, but not discriminating investment. It is more or less realistic, though existing tax relieves for almost-dead enterprises violate such constraint, may be. If included in optimization, the constraint can be expressed also as $(x^I, y^I) = (s{*}x^I, t{*}y^I)$, $(s,t) \le (1,1)$, where (s,t) denote tax relieves or tax-holiday present values.

We can suppose some lag $L \ge 0$ for new investments to become capital, and expect related revenue only starting from *L-th* year after investment. However, simultaneously we should take into account yesterday's investments (already being implemented) and summarize them until L. Therefore, it is not a great simplification to suppose $L = 0$ insofar.

To construct the needed utility function, we should express tradeoff between current and future goals. We shall take usual "future utility" concept, introduced in the form of discounted infinite sum of temporary utility gains, with a discount rate δ - patience factor (say, $\delta = 0.95$). As previously, we shall attach weight w_1 to tax-revenue goal of n-th year $Rev(n,x,y)$ and weight w_2- to production/employment goal $Kap(n,x,y)$ of n-th year, to form the objective function:

$$U(x,y) = \sum_{n=0}^{T} \delta^{(n)} \left(w_1\, Rev(n,x,y,xs,yt) + w_2\, Kap(n,x,y,xs,yt)\right).$$

Now we should substitute $Rev(.)$ and $Kap(.)$ in the objective function by exact formulae for capital and revenue, to convert dynamic problem into pseudo-static one.

To do this, we should distinguish revenue from old capital K^0 *or* (K^w, K^s) and revenue from new capital, because different taxes may be applied to them (the promised now holidays must operate in future). All new capital will operate under new taxes denoted further $\sigma = sx, x = ty$, which describe present value of tax holidays .[14] Stable taxation we suppose.

Our expected present value of future (n=1,2,...) stream of revenue from the **old** (born before 1999 and older, if we are deciding in 1999) capital, is based on the declining, due to depreciation, capital sequence $a(K^w\,\mathbf{K}^w(x,y) + K^s)$, $a^{(2)}(K^w\,\mathbf{K}^w(x,y) + K^s)$, $a^{(3)}(K^w\,\mathbf{K}^w(x,y) + K^s)$,.... We can express this utility stream gained from weak and strong old capital until year T as:

$$U^{T(old)}(x,y) = \sum_{n=0}^{T} \delta^{(n)} a^{(n)} \left[\mathrm{w}_1(K^w \boldsymbol{R}^{Kw}(x,y) + K^s \boldsymbol{R}^{Ks}(x,y)) + \mathrm{w}_2(K^w\,\mathbf{K}^w(x,y) + K^s)\right],$$

where $(K^w \boldsymbol{R}^{Kw}(x,y) + K^s \boldsymbol{R}^{Ks}(x,y)) = [K^w\, [\,(1-qp-qx-y+qyx)/(1-y)[(x - X^{fed}) +$
$+ (y - Y^{fed}) \cdot ((1/2 + ((1-qp-qx-y+qyx)/(1-y))/2 - xq)/q)] + (y - Y^{fed})C^K] +$
$+ K^s\, [\,(x - X^{fed}) + (y - Y^{fed})\, 0.5\, (\pi + x + 1/g)(1 - g\,\pi - g\,x) + (y - Y^{fed})C^I]].$

[13] This can happen when, in contrast with the stable-strategy setting, when the legislator is short-sighted or too much impatient to get tax revenue, for whatever reasons. Or, maybe, rapid growth or rapid decline of a region forces legislator to solve dynamic-dynamic rather then stable-dynamic problem.

[14] Of cause, it borns some bias in the below estimate: present value of eternal present-value tax relief does not quite correctly reflect the loss of revenue for region from tax holidays. We suppose it not too high.

Somewhat differently we should express our utility stream from young capital.

Stable taxation under the assumed stable circumstances implies also stable expected investment $I = I^m \boldsymbol{I}(\sigma, x)$. We estimate current younger-then-1999 capital stock of a future year n as $K^{n(new)} = aK^{n\text{-}1(new)} + I$. Using standard progression formula we get new capital $K^{n(new)} = \sum_{k=1}^{n} a^{(k)} I = I*(a^{(n+1)} - a)/(a\text{-}1) = I^m \boldsymbol{I}(\sigma, x)*(a^{(n+1)} - a)/(a\text{-}1)$.

So, we express our present value of future utility, gained from young capital until year T as

$U^{T(young)}(\sigma, x) = \sum_{n=1}^{T} \delta^{(n)} (a^{(n+1)} - a)/(a\text{-}1)\ [\mathrm{w}_1 I^m \boldsymbol{R}^I(\sigma, x) + \mathrm{w}_2 I^m \boldsymbol{I}(\sigma, x)] =$

$= \sum_{n=1}^{T} \delta^{(n)} [\mathrm{w}_1 I^m *(a^{(n+1)} - a)/(a\text{-}1)]* \boldsymbol{I}(\sigma, x)\ [\ [\sigma - X^{fed} + (x - Y^{fed})\cdot((1/2 + \boldsymbol{I}(\sigma, x)/2 -$

$- \sigma\ g)/g)] + (y - Y^{fed}) C^I + \mathrm{w}_2].$

Now, combining the above expressions, supposing infinite horizon $T=\infty$, we can express the objective function as

$U(x,y) = U^{T(old)}(x,y) + U^{T(young)}(\sigma, x) =$

$= \sum_{n=0}^{T} \delta^{(n)} a^{(n)} [\mathrm{w}_1(K^w \boldsymbol{R}^{Kw}(x,y) + K^s \boldsymbol{R}^{Ks}(x,y)) + \mathrm{w}_2(K^w \mathbf{K}^w(x,y) + K^s)] +$

$+ \sum_{n=0}^{T} \delta^{(n)} (a^{(n+1)} - a)/(a\text{-}1)\ [\mathrm{w}_1 I^m \boldsymbol{R}^I(\sigma, x) + \mathrm{w}_2 I^m \boldsymbol{I}(\sigma, x)] =$

$= 1/(1 - \delta a) * [\mathrm{w}_1(K^w \boldsymbol{R}^{Kw}(x,y) + K^s \boldsymbol{R}^{Ks}(x,y)) + \mathrm{w}_2(K^w \mathbf{K}^w(x,y) + K^s)] +$

$+\ a/(a-1) * [1/(1 - \delta a) - 1/(1 - \delta)][\mathrm{w}_1 I^m \boldsymbol{R}^I(\sigma, x) + \mathrm{w}_2 I^m \boldsymbol{I}(\sigma, x)].$

That means utility function having following weights: weight $v_1 = w_1/(1 - \delta a)$ for revenue depending upon taxes (x,y) (“old-capital revenue”), weight $v_2 = w_2/(1 - \delta a)$ for related capital (“old-type capital”), weight $v_3 = w_1\ a/(a-1) * [1/(1 - \delta a) - 1/(1 - \delta)]$ for revenue depending upon taxes (σ, x) (“future young-capital revenue”), weight $v_4 = w_2\ a/(a-1) * [1/(1 - \delta a) - 1/(1 - \delta)]$ for related capital (“future young capital”).

Thus we have related our multi-period objective function to some pseudo-static or, rather, two-period function with weights (v_1, v_2, v_3, v_4) weighing “today” revenue and capital depending upon (x,y) with “tomorrow” revenue and capital depending upon (σ, x). We optimize it as if today capital is supposed to disappear tomorrow.

If we assume the constraint on tax relieves $(\sigma, x) \leq (x,y)$ not binding, then optimization of this function can be decomposed into 2 separate problems (dropping constant multipliers) that happens to be just static problems:

$U^{T(old)}(x,y)\ \text{->}\ max_{x,y}$

$U^{T(young)}(\sigma, x)\ \text{->}\ max_{\sigma, x}$.

So, impatience δ and depreciation a play no role when tax holidays are essentially used!

Take more general case, including the case when tax relieves or holidays are unavailable: $(\sigma, x) = (x,y)$. Then impatience and depreciation factors play some role in the resulting total pseudo-static or 2-period objective function, similar to the case, when all current capital becomes obsolete during a year, while all next-year capital results from investment:

$U(x,y, \sigma, x) = U^{T(old)}(x,y) + U^{T(young)}(\sigma, x) =$

$= \mathbf{v}_1(K^w \boldsymbol{R}^{Kw}(x,y) + K^s \boldsymbol{R}^{Ks}(x,y)) + \mathbf{v}_2(K^w \mathbf{K}^w(x,y) + K^s) +$

$+ \mathbf{v}_3 I^m \boldsymbol{R}^I(\sigma, x) + \mathbf{v}_4 I^m \boldsymbol{I}(\sigma, x),$

where variables $(\sigma, x) = (xs, yt)$,

coefficients $\mathbf{v}_1 = w_1/(1-\delta a)$, $\mathbf{v}_2 = w_2/(1-\delta a)$,

$\mathbf{v_3} = w_1\, a/(a-1)^* [1/(1-\delta a) - 1/(1-\delta)]$,

$\mathbf{v}_4 = w_2\, a/(a-1)^* [1/(1-\delta a) - 1/(1-\delta)]$.

However, it will be more convenient for us to normalize further these coefficients, it makes no difference for optimization:

$$v_1 = w_1, \quad v_2 = w_2, \quad v_3 = w_1 C_{coeff}, \quad v_4 = w_2\, C_{coeff},$$

$$C_{coeff} = a/\,(a-1)\,/(1-\delta a)^* [1/(1-\delta a) - 1/(1-\delta)].$$

How large can be C_{coeff} ? For instance, $C_{coeff}(a{=}0.9, \delta{=}0.5) = 2.97521$,

$C_{coeff}(a{=}0.9, \delta{=}0.6) = 6.37996$, $C_{coeff}(a{=}0.9, \delta{=}0.7) = 15.3397$, thus even with rather low patience 0.5, weight v_3 of future revenue in objective function is substantial.

For the below Theorem 2 we shall optimize this function w.r.t. x, supposing tax relieves or tax holidays unavailable: $(s,t) = (1,1)$, so $(\sigma, x) = (x,y)$. The function is a power-2 polynomial w.r.t. x, like $Ax^{(2)}+Bx+H$. These factors A,B,H we should express now in initial terms.

Into new function $U(x,y,xs,yt)$ we should substitute the earlier expressions for revenue, capital and investment (denoting $p^K = p$ further), thus we obtain the below package of

formulae for pseudo-static optimization:

$$K^{(old)}(x,y) = (K^w\, \mathbf{K}^w(x,y) + K^s\,), \quad \mathbf{K}^w(x,y) = (1-q\,p-q\,x-y+g\,y\,x)/(1-y),$$
$$R^{(old)}(x,y) = K^w\, \boldsymbol{R}^{Kw}(x,y) + K^s\, \boldsymbol{R}^{Ks}(x,y),$$
$$\boldsymbol{R}^{Ks}(x,y) = (x-X^{fed}) + (y-Y^{fed})\, 0.5\, (\pi - x + 1/g)(1-g\,\pi - g\,x) + (y-Y^{fed})C^I, \tag{1-1}$$
$$\boldsymbol{R}^{Kw}(x,y) = (1-q\,p - q\,x - y + q\,y\,x)/(1-y)[(x-X^{fed}) + (y-Y^{fed})\cdot((1/2+$$
$$+((1-q\,p-q\,x-y+q\,y\,x)/(1-y))/2-xq)/q)] + (y-Y^{fed})C^K,$$
$$K^{(young)}(xs,yt) = I^m\, \boldsymbol{I}(xs,yt) = I^m\,(1-g\,\pi - g\,xs - yt + g\,yt\,xs)/(1-yt),$$
$$R^{(young)}(xs,yt) = I^m\, \boldsymbol{R}^I(xs,yt) = I^m\,((1-g\,\pi - g\,xs - yt + g\,yt\,xs)/(1-yt)[(xs-X^{fed}) +$$
$$+(yt-Y^{fed})\cdot((1/2 + ((1-g\pi - g\,xs - yt + g\,yt\,xs)/(1-yt))/2 - xsg)/g)] + (yt-Y^{fed})C^I),$$

Subject to budget constraint $(K^w\, \boldsymbol{R}^{Kw}(x,y) + K^s\, \boldsymbol{R}^{Ks}(x,y)) \geq G$, and to legal bounds on (x,y), we optimize function

$$U(x,y,xs,yt) = \quad U^{T(old)}(x,y) + U^{T(young)}(xs,yt) = \tag{1-2}$$
$$= w_1(K^w\, \boldsymbol{R}^{Kw}(x,y) + K^s\, \boldsymbol{R}^{Ks}(x,y)) + w_2(K^w\, \mathbf{K}^w(x,y) + K^s) +$$
$$+\; v_3\, I^m\, \boldsymbol{R}^I(xs,yt) + v_4\, I^m\, \boldsymbol{I}(xs,yt) =$$
$$= w_1\, K^w\, \mathbf{(}\, (1-q\,p - q\,x - y + q\,y\,x)/(1-y)[(x-X^{fed}) + (y-Y^{fed})\cdot((1/2+$$
$$+((1-q\,p-q\,x-y+q\,y\,x)/(1-y))/2-xq)/q)] + (y-Y^{fed})C^K\,\mathbf{)} +$$
$$+\, w_1\, K^s\, \mathbf{(}\, (x-X^{fed}) + (y-Y^{fed})\, 0.5\, (\pi - x + 1/g)(1-g\,\pi - g\,x) + (y-Y^{fed})C^I\,\mathbf{)} +$$
$$+\, w_2\, \mathbf{(}\, K^w\,(1-q\,p-q\,x-y+g\,y\,x)/(1-y) + K^s\,\mathbf{)} +$$
$$+\, v_3\, I^m\, \mathbf{(}(1-g\,\pi - g\,xs - yt + g\,yt\,xs)/(1-yt)[(xs-X^{fed}) +$$
$$+(yt-Y^{fed})\cdot((1/2 + ((1-g\pi - g\,xs - yt + g\,yt\,xs)/(1-yt))/2 - xsg)/g)] + (yt-Y^{fed})\, C^I\,\mathbf{)} +$$
$$+v_4\, I^m\,(1-g\,\pi - g\,xs - yt + g\,yt\,xs)/(1-yt) \quad = \quad Ax^{(2)} + B\,x + H,$$

where

$$H = (2*K^s*w_2 - 2*K^s*w_1*X^{fed} + 2*C^I*I^m*v_3*(y - Y^{fed}) + \qquad \textbf{(1-3)}$$
$$+ 2*(C^I*K^s + C^K*K^w)*w_1*(y - Y^{fed}) +$$
$$+ K^s*(g^{(}-1) + \pi)*(1 - g*\pi)*w_1*(y - Y^{fed}) +$$
$$+ (K^w*(-1 + p*q + y)*(2*w_1*(-1 + y)*(y - Y^{fed}) +$$
$$+ q*(2*w_2*(-1 + y) + w_1*(2*X^{fed} + p*y - 2*X^{fed}*y -$$
$$- p*Y^{fed}))))/ (q*(-1 + y)^{(2)}) + (I^m*(-1 + g*\pi + y)* (2*v_3*(-1 + y)*(y - Y^{fed}) +$$
$$+ g*(2*v_4*(-1 + y) + v_3*(2*X^{fed} + \pi*y - 2*X^{fed}*y -$$
$$- \pi*Y^{fed}))))/ (g*(-1 + y)^{(2)}))/2,$$

$$B = (2*K^s*w_1*(-1 + y)*(1 - y + Y^{fed}) + \qquad \textbf{(1-4)}$$
$$+ K^w*(-2*w_1 + 2*p*q*w_1 + 2*q*w_2 - 2*q*w_1*X^{fed} + 7*w_1*y - 4*p*q*w_1*y -$$
$$- 2*q*w_2*y + 2*q*w_1*X^{fed}*y - 5*w_1*y^{(2)} + w_1*(-5 + 4*p*q + 5*y)*Y^{fed}) +$$
$$+ I^m*(-2*g*v_4*(-1 + y) +$$
$$+ v_3* (-2 + 2*g*\pi - 2*g*X^{fed} + 7*y - 4*g*\pi*y + 2*g*X^{fed}*y - 5*y^{(2)} +$$
$$+ (-5 + 4*g*\pi + 5*y)*Y^{fed})))/(2*(-1 + y)),$$

$$A = 0.5 [g*(I^m*v_3*(-2 + 3*y - 3*Y^{fed}) + K^s*w_1*(y - Y^{fed})) + \qquad \textbf{(1-5)}$$
$$+ K^w*q*w_1*(-2 + 3*y - 3*Y^{fed})] .$$

In the next subsection we shall discuss optimization of the obtained objective function w.r.t. (x,y), supposing fixed $(s,t) \leq (1,1)$. The corresponding Lagrangian will be:

$$L(x,y, l_1, l_2, m_3, m_4) = \qquad \textbf{(1-6)}$$
$$= w_1(K^w \boldsymbol{R}^{Kw}(x,y)+ K^s \boldsymbol{R}^{Ks}(x,y)) + w_2(K^w \mathbf{K}^w(x,y)+ K^s) +$$
$$+ v_3 I^m \boldsymbol{R}^I(xs,yt) + v_4 I^m \boldsymbol{I}(xs,yt) +$$
$$+ l_1(xs-X^{min}) - l_2(x-X^{max})+ m_3 (yt-Y^{min}) - m_4(y-Y^{max})+ \rho (K^w \boldsymbol{R}^{Kw}(x,y)+ K^s \boldsymbol{R}^{Ks}(x,y) - G),$$

where $(l_1, l_2, m_3, m_4 , \rho) \geq 0$ are the Lagrange multipliers of the constraints.

Remark about stable-capital (*equilibrium-point*) *optimization setting*:

Above we have compared dynamics of old and new capital. But there can be an alternative setting, based only on new capital, totally neglecting short-term considerations connected with existing capital stock. Suppose, that the decision-maker have in mind only to place *sometimes in far future* the region into a stable situation (equilibrium), where the taxation policy and all variables are stable for a long period. We should assume $(x,y) = (xs,yt) = (x^I,y^I)$, since tax rates and tax relieves promised to investors will actually operate always (recall, we convert also tax holidays into present-value-equivalent tax relieves).

Stability means, that annual gross investments exactly outweigh the annual depreciation (denoted a>0), so that capital K would remain the same annually: $K=I/(1-a)$. We again suppose, that we have stable investment curve parameters I^m,g,C^I,π. They should be supposed equal to parameters $K^m/(1-a),g,C^K,\pi^K$ of "strong capital" curve, which should be supposed to be a triangle, not trapeze, in this setting.

So, we come to somewhat different then earlier 2-tax stable-dynamic (in essence pseudo- static) optimization problem; subject to constraint $I^m/(1-a)* \boldsymbol{R}^I(x,y) \geq G$, we should optimize the function:

$$U(x,y) = \sum_{t=0}^{T} \delta^{(t)}[w_1(K^w \boldsymbol{R}^{Kw}(x,y)+ K^s \boldsymbol{R}^I(x,y)) + w_2(K^w \mathbf{K}^w(x,y)+ K^s \boldsymbol{I}(x,y)) +$$
$$+ w_1 I^m \boldsymbol{R}^I(x,y) + w_2 I^m \boldsymbol{I}(x,y)],$$ that is the same as to optimize
$$U(x,y) = w_1(\mathrm{d}\, I^m/ (1-a)*\boldsymbol{R}^{Kw}(x,y)+ I^m/ (1-a)* \boldsymbol{R}^I(x,y)) + w_2(\mathrm{d}\, I^m/ (1-a)* \mathbf{K}^w(x,y)+$$
$$+ I^m/ (1-a)* \boldsymbol{I}(x,y)) + w_1 I^m \boldsymbol{R}^I(x,y) + w_2 I^m \boldsymbol{I}(x,y),$$

(we have used here parameter 0< d <1, describing regular share of "weak" operating capital to strong capital). One can see no principal difference from static problem in this setting.

In general, we suppose this "equilibrium" reasoning to be less characteristic for impatient Russian decision-makers, then the previous approach, so we do not use it further.

1.6. *Analytical and numerical study of optimization problem*

For our stable-dynamic (pseudo-static) 2-tax optimization problem we have constructed the Lagrangian with primal variables (x,y), dual variables (ρ , l_i , m_i), and fixed parameters $(s,t)\leq(1,1)$. We can reformulate it now dropping unimportant constants and installing expressions as:

$$
\begin{aligned}
&L(x,y,\rho,l_1,l_2,m_3,m_4) = \qquad\qquad \textbf{(1-7)}\\
&= (w_1+\rho)(K^w \boldsymbol{R}^{Kw}(x,y)+K^s \boldsymbol{R}^{Ks}(x,y)) + w_2(K^w \mathbf{K}^w(x,y)+K^s) +\\
&\quad + v_3 I^m \boldsymbol{R}^I(xs,yt) + v_4 I^m \boldsymbol{I}(xs,yt) + l_1 xs - l_2 x + m_3\ yt - m_4 y =\\
&= (w_1+\rho)*(x-X^{fed})K^s+(w_1+\rho)*(y-Y^{fed})K^s*1/2(\pi-x+1/g)(1-g\,\pi-g\,x)+\\
&\quad + (w_1+\rho)*(y-Y^{fed})(K^s C^I+K^w C^K)+\\
&\quad +K^w(1-q\,p-q\,x-y+q\,y\,x)/(1-y)*((w_1+\rho)*(x-X^{fed}+\\
&\quad + (y-Y^{fed})*((1/2+1/2\,(1-q\,p-q\,x-y+q\,y\,x)/(1-y)-q\,x)/q))+w_2)+w_2 K^s +\\
&+I^m(1-g\,\pi-g\,xs-yt+g\,yt\,xs)/(1-yt)*(v_3(xs-X^{fed}+(yt-Y^{fed})*((1/2+\\
&+1/2\,(1-g\,\pi-g\,xs-yt+g\,yt\,xs)/(1-yt)-g\,xs)/g))+v_4)+v_3*(yt-Y^{fed})I^m C^I +\\
&+ l_1 x\,s - l_2 x + m_3\ y\,t - m_4 y\,.
\end{aligned}
$$

Remark for special (*static*) *case*: For the case when tax holidays are possible, the problem, as shown above, split into 2 separate problems. The second one, i.e. optimizing (s,t) for fixed (x,y) is equivalent to optimizing (x,y) under assumption $(s,t)=(1,1)$ and $((w_1+\rho),w_2)$ being infinitely smaller then (v_3,v_4), that means that only future goals matter. Then we come to much simpler Lagrangian with similar properties:

$$
\begin{aligned}
&L_2(x,y,l_1,l_2,m_3,m_4) =\\
&= I^{max}(1-g\,\pi-g\,x-y+g\,y\,x)/(1-y)*(v_3*(x-X^{fed}+(y-Y^{fed})*((0.5+\\
&\quad +0.5(1-g\,\pi-g\,x-y+g\,y\,x)/(1-y)-g\,x)/g))+v_4) + (l_1-l_2)x+(m_3-m_4)\,y.
\end{aligned}
$$

General Lagrangian function, and simplified one, and function $U(x,y)$, all appear to be rational functions with 3-order polynomial in the nominator (with second order w.r.t. x), and with linear denominator. The first-order conditions (gained by computer) are (let $(s,t)=(1,1)$, $v_1:=(w_1+\rho)$ further):

$$
\begin{aligned}
&D_xL= (2*K^s*v_1*(-1+y)*(1-y+g*x*y+Y^{fed}-g*x*Y^{fed}) +\\
&\quad +K^w*v_1(-2+2*p*q+2*q*w_2/(w_1+\rho)+4*q*x-2*q*X^{fed}+7*y-\\
&\quad -4*p*q*y-2*q*w_2*y/(w_1+\rho)-10*q*x*y+2*q*X^{fed}*y-5*y^{(2)}+\\
&\quad +6*q*x*y^{(2)}+(-5+4*p*q+6*q*x+5*y-6*q*x*y)*Y^{fed}) +\\
&\quad + I^m*(-2*g*v_4*(-1+y) +\\
&+ v_3*(-2+2*g*\pi+4*g*x-2*g*X^{fed}+7*y-4*g*\pi*y-10*g*x*y+\\
&\quad +2*g*X^{fed}*y-5*y^{(2)}+6*g*x*y^{(2)}+\\
&\quad + (-5+4*g*\pi+6*g*x+5*y-6*g*x*y)*Y^{fed})))/(2*(-1+y)) +(l_1-l_2) = 0
\end{aligned}
$$

$$
\begin{aligned}
&D_yL= (K^s*q*v_1*(-1+y)^{(3)} - g^{(2)}*K^s*q*v_1*(\pi-x)*(\pi+x)*(-1+y)^{(3)} +\\
&\quad + g*(-2*K^w*v_1-2*C^I*K^s*q*v_1-2*C^K*K^w*q*w_1+3*K^w*p*q*v_1-\\
&\quad - K^w*p^{(2)}*q^{(2)}*v_1+2*K^w*p*q^{(2)}*v_2+2*K^s*q*v_1*x+5*K^w*q*v_1*x-
\end{aligned}
$$

$- 2*K^w*p*q^{(2)}*v_1*x - 3*K^w*q^{(2)}*v_1*x^{(2)} - 2*K^w*p*q^{(2)}*v_1*X^{fed} -$
$-(-6*K^w*v_1 - 6*C^I*K^s*q*v_1 - 6*C^K*K^w*q*v_1 + 3*K^w*p*q*v_1 +$
$+ K^w*p^{(2)}*q^{(2)}*v_1 + 2*K^w*p*q^{(2)}*w_2 + 6*K^s*q*v_1*x + 15*K^w*q*v_1*x -$
$- 2*K^w*p*q^{(2)}*v_1*x - 9*K^w*q^{(2)}*v_1*x^{(2)} - 2*K^w*p*q^{(2)}*v_1*X^{fed})*y -$
$- 3*v_1*(2*(K^w + C^I*K^s*q + C^K*K^w*q) - (2*K^s + 5*K^w)*q*x +$
$+ 3*K^w*q^{(2)}*x^{(2)})*y^{(2)} + v_1*(2*(K^w + C^I*K^s*q + C^K*K^w*q) -$
$- (2*K^s + 5*K^w)*q*x + 3*K^w*q^{(2)}*x^{(2)})*y^{(3)} +$
$+ K^w*p*q*v_1*(2*p*q - (-3 + 4*q*x)*(-1 + y))*Y^{fed}) +$
$+ I^m*q*(-2*g^{(2)}*\pi*v_4*(-1 + y) +$
$+ v_3*(-2 - 2*C^I*g + 3*g*\pi - g^{(2)}*\pi^{(2)} + 5*g*x - 2*g^{(2)}*\pi*x -$
$- 3*g^{(2)}*x^{(2)} - 2*g^{(2)}*\pi*X^{fed} + 6*y + 6*C^I*g*y - 3*g*\pi*y -$
$- g^{(2)}*\pi^{(2)}*y - 15*g*x*y + 2*g^{(2)}*\pi*x*y + 9*g^{(2)}*x^{(2)}*y +$
$+ 2*g^{(2)}*\pi*X^{fed}*y - 6*y^{(2)} - 6*C^I*g*y^{(2)} + 15*g*x*y^{(2)} -$
$- 9*g^{(2)}*x^{(2)}*y^{(2)} + 2*y^{(3)} + 2*C^I*g*y^{(3)} - 5*g*x*y^{(3)} + 3*g^{(2)}*x^{(2)}*y^{(3)} +$
$+ g*\pi*(3*(-1 + y) + 2*g*(\pi + 2*x - 2*x*y))*Y^{fed})))/ (2*g*q*(-1 + y)^{(3)})+$
$+ (m_3-m_4) = 0.$

Generally speaking, we should solve now this system w.r.t. (x,y), finding the solutions $X^{opt}(w_i,p,l_i,m_i,...)$,$Y^{opt}(w_i,p,l_i,m_i,...)$, and (l,m) that fit boundary constraints, supplementary slackness constraints and concavity conditions. Thus we could determine[15] the exact solution (X^{opt}, Y^{opt}) which is really a global maximum within the constrains. The solution would stand for couple of optimal-tax functions (X^{opt},Y^{opt}):$R^{12} \to R^2$; depending upon all parameters $(I^m, g, \pi, w_1, w_2, C^I, K^s, K^w, C^K, Y^{fed}, X^{fed}, Y^{min}, Y^{max}, X^{min}, X^{max})$.

However, direct symbolic solving this rational system of 2 equations even without additional constraints turns out to be impossible: too tedious, both for manual and for machine calculations. We have used *Mathematica*-3.0 software, supposing it to be the most powerful of the kind, but it failed in symbolic and even in numeric solving the system of equations. The point is that the objective function U has unpleasant form (see plots in Appendix) being generally non-concave on R_+^2, and even on [0,1]*[0,1]. However, when y is fixed, it has 1 maximum w.r.t. x (it is a parabola in this direction), for "most natural" parameters. This maximum, combined with boundary conditions, will be used below for one-tax game.

In another direction, varying y and fixing x on [0,1], the objective function has 3 critical points and 1 discontinuity point (y=1) , since it has always zigzag shape (see the plots) in "y" direction. Fixing parameter x, we have found these points $y^1(x)$,$y^2(x)$,$y^3(x)$ by exact computer solving the equation $D_yL(y,x) = 0$. Only one of these 3 critical points (the smallest "y") is a local maximum fitting natural constraints $0 \le y < 1$. *W*ithin these constraints there is also local minimum around y=0.95 (for realistic parameters), while for larger y the function increases infinitely until y=1. So, if we believe, that upper legal bound on profit tax is always less then, say, 0.9 (in Russian law it is 0.35, but including VAT it becomes near 0.60), then the objective function is concave in "y" direction on

[15] By comparing roots of equations with other constraints, manipulating if necessary with dual variables *l*, *m*, and checking concavity.

admissible zone. Yet the concavity along abscissa and along ordinate is necessary, but not sufficient to guarantee concavity in general on the admissible part of R^2. In contrast, continuity on this zone is obvious, guaranteeing maxima.

To exclude several isolated maxima, we should establish general concavity or quasi-concavity. It is a hard task, since formulae are huge, preventing exact operation. However, they were used for numeric solving. Machine experiments with plotting function $U(x,y)$ showed, that general shape of this surface remains more or less the same with respect to our numerous parameters, only the slope and peaks move. This is confirmed by the type of rational function $U(...,x,y)$ as described above (power 3 in the nominator).

We performed extensive numerical investigation of concavity, exhausting with rather dense net of tests the whole domain of "realistic" exogenous parameters $K, pi, g, q,...$ and a lot of (x,y) combinations (see the report in Appendix). Function $U(...,x,y)$ was found to be *concave on the realistic admissible area* $[0,X^{max}]*[Y^{min},Y^{max}]$, at least we have not found a counterexample. So, under reasonable parameters (in particular, when $y\text{-}Y^{fed} < 2/3$, $d>0.1$) we may suppose it concave with high probability. Hence, $U(x,y)$ as well as $L(x,y)$, may be expected to have single maximum with respect to both variables on this narrow domain. It is worth noting that this result used value $d= K^w / K^s =0.1$, in contrast with $d=0$ considered in subsection 2.2. This explains the difference: short-term revenue objective function may be non-concave under 2.2.

Having understood the optimization problem structure, we tried to reveal character of solutions, dependent on parameters. We have explored with computer several series of *numeric* solutions to *special case* of our problem (using dyhotomia algorithm and random algorithm), namely the case related to optimizing only tax holidays, that is taking $K^s=K^w=0$ in our objective function. We tried to exhaust again the whole area of admissible ("reasonable") numerical values of parameters, with more or less dense net of tests. The Appendix explains what we suppose empirically "reasonable" or "realistic".

Namely, we studied two packages of intervals: with "low" taxes, and with "high" taxes.

The first is $0.05<pi<0.18$, $0\le w_1\le 1$, $0.8\le g\le 3.2$, $0=X^{min}$, $X^{max} =0.02$, $0.13=Y^{min}$, $Y^{max} =0.35$, $Y^{fed}=0.13$, chosen directly from the Russian law. The results may be interesting for some other country, or for realizing the mathematical structure of the problem itself. However, for Russia we realized that other taxes, first of all VAT, essentially influence the solutions. So, we roughly estimated, how much profit tax and property tax is enough to have the same impact on hampering investment as the three taxes altogether, plus secondary taxes (see Appendix). This born a "high-tax" seria with doubled tax parameters: $p=0.05$, $p=0.09$, $0\le w_1\le 1$, $2.0\le g\le 4.0$, $0.02=X^{min} \le x\le X^{max} =0.04$, $0.26=Y^{min}\le y\le Y^{max} =0.70$, $Y^{fed}=0.26$ (we looked briefly also on a broader range of parameters, with the same results, but less thoroughly).

Low-tax series) Dividing the domain of each of the 3 parameters p,w_1,g into 10 intervals, we tried each of 1000 points. We were plotting our objective function U on the admissed rectangle, and the solutions $Xopt(y^{const}),Yopt2(x^{const})$ to 2 separate equations $DL_x(x,y^{const}) =0$, $DL_y(x^{const},y) =0$. We plotted also 2-dimensional functions $U(Xopt(y),y),U(x,Yopt2(x))$ related to $(Xopt, Yopt)$ reply-functions, and 2-dimensional functions $U(x^{const},y),U(x,y^{const})$ related to the boundaries of our rectangle domain (where x^{const},y^{const} were taken as boundary values (0.0,0.02,0.13,0.35)). In addition we plotted "general graphs" gathering *critical points* of preference-for revenue parameter w_1, where the taxation strategy switches from one-tax to two-tax form, these graphs are more informative, and checked the optima by an heuristic optimizing program (see Appendix for pictures and details).

Results. Most tests of this seria (and of the next also) have shown only *boundary solutions*, never attaining optimum in interiority of the admissible rectangle ([0.0,0.02], [0.13,0.35]). In Appendix "General graphs" one can see, that with small preference for revenue w_1, in all tests with "reasonable" parameters the property tax becomes positive only when profit tax hits its upper limit, whatever it can be. Even if we allow property tax to be negative by setting negative lower bound on it (like -1,-10,-100), this bound is reached in optimal solution, when we solve for (x,y) within $0< y< 1$, since the U surface is a usually sloped ascending to left far corner (see plots). So, usually a **rational legislator will chose profit tax** Y until possible, and **only constraints force him to use property tax** X. This common rule is violated in the region $0.12 < p$, $3.5<g$, $0.98<w_1$, $0.13 < p$, $3.4< g$, $0.97< w_1$, ..., $0.18< p$, $2.4< g$, $0.95< w_1$. However, only the last of these 6 exceptional graphs showing really interior maximums, as shown in another Appendix. All these tests used unreasonably high profit p^k and high revenue preference w_1. So the rule can be supposed mostly holding.

High-tax series) Exactly the same rule was found for the series of tests with parameters p=0.05, p=0.09, $0< w_1< 1$, $2.0< g< 4.0$, $0.02=X^{min} <x< X^{max} =0.04$, $0.26=Y^{min}<y< Y^{max} =0.70$, Y^{fed}=0.26. Most expressive examples of this series see in Appendix "Direct optimization". Tests showed that, again, profit tax would be preferred when possible, except for narrow and unrealistic range of parameters, where property tax prevails. For instance, Test 3: X^{min}=0.00; X^{max}=0.04; Y^{min}=0.40; Y^{max}=0.70; X^{fed}=0.005, g=2.3; p=0.09; w_1>0.90 (too small bound on property tax, too high preference for revenue w_1 and high alternative profit p). Test4 - with X^{min}=0.00; X^{max}=0.04; Y^{min}=0.40; Y^{max}=0.70; X^{fed}=0.005; Y^{fed}=0.26; g=3.6; w_1=0.96; - too high higher sensitivity factor; this test is the *lowest* (g=3.6) to show *non-trivial two-tax solutions*, while with higher g non-triviality also happens.

In all tests, the natural dependencies are present:

- increasing slope g of the investment curve (sensitivity to taxes) always drives down the optimal taxes;
- increasing preference for revenue w_1 (as opposed to preference for capital/employment) always drives taxes up;

Non-trivial findings are that:

- profit tax is usually preferred to property tax, being less harmful to investment for the same amount of revenue, only legal bounds on profit tax may force usage of property tax;
- this is not the case when alternative profit p, sensitivity factor g and preference for revenue w_1 are all "very high" – then composite taxes or even preference for property tax become possible.

To close the section, note, that our scheme of analysis (but not the numerical results) can be applied not only to capital market, but also to various markets with a linearized demand curve and with flat supply curve. Property tax is analogue of “unit tax”, while profit tax is like “ad valorem tax”, their optimal mixture is found. However, formulae for demand dependence upon taxes and for revenue are slightly different.

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2. TAX COMPETITION GAME

We attempt to develop theoretical framework for arguing about possible outcomes of the interregional tax competition in Russia, to distinguish and estimate main parameters yielding *divergence* effect, that means increasing asymmetry of development. Another question is the efficiency and welfare implications of divergence. The below models use the estimates and ideas obtained in E.Kolomak’s empirical analysis, and the above tax-optimization results.

2.1. General scheme of dynamic tax competition game

Relying on the above rational-behavior model, now we outline the dynamic agents-interaction model.

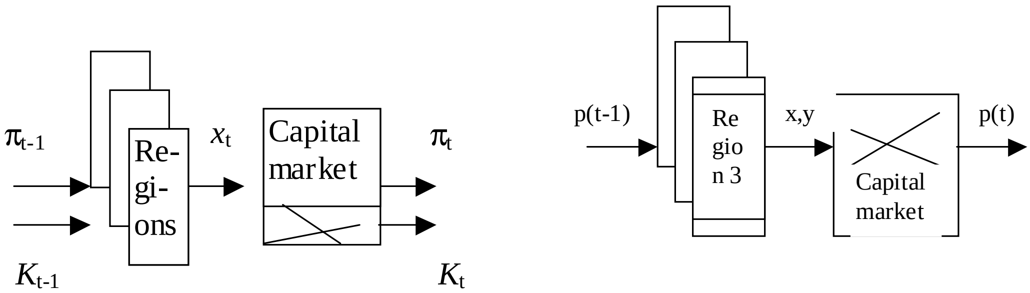

Figure 2. *Game sequence*

Game concept. Slightly detailing the concept explained in the introduction, we describe now tax competition in terms of infinite sequential game with 91 players: 89 regions, dummy player “capital market”, and dummy “consequences of temporary equilibrium”. They act in turn. The regions simultaneously establish their tax rates, then relying on these rates capital market equilibrate demand and supply for capital (in usual Marshalian-cross manner), yielding equilibrium profit rate, then investments and all other consequences including capital stock and tax revenues are calculated. Then the whole play repeats. The regions are supposed to behave myopically, taking profit rate as given, in this relation they are price-takers. So, if the process converges to some equilibrium, it is a Nash equilibrium among 91 players. Investors, entrepreneurs – all are beyond the scene, being described by some functions of demand and supply.

We aim to show the mechanism of effects arising due to tax competition, namely divergence and efficiency. Let us discuss expected *performance* of the outlined game, including additional elements, like public goods, not investigated in present version.

The principal scheme of “divergence” possibility in taxation models

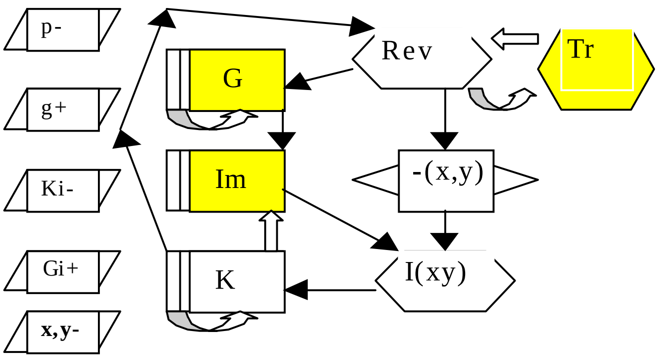

Figure 3: *The scheme explaining possible ``divergence effect” in the taxation models*. Positive feedback circles are black, clock-directed. Negative feedback circles are white, anti-clock-directed. Shaded are the elements skipped in this version of the model. Exogenous parameters are marked with “+” if their large value enforce divergence, or “-“ in the opposite case.

General scheme of possible regional- taxation models describing “divergence” possibility see on Fig.3. It is the scheme of a process, born by the game discussed. Main exogenous parameters are: alternative (outside this region) profit rate *p*, sensitivity-to-taxes parameter *g* (slope of demand for capital), initial capital stock *Ki*, initial public goods stock *Gi*, legal regulations of taxation sphere, expressed in admissible minimal and maximal values of taxes **x,y**, in transfer parameters. Endogenous stock- variables are squares, two of them include negative feedback: capital depreciation. These are accumulated public goods volume *G* and accumulated capital volume *K*. Another stock-type variable is the ``maximal-possible investment” parameter I^m, describing investment possibilities, which can be exhausted by investment, but restored and enhanced by public goods volume (in the simplest version it is constant). There are 4 flow-variables. Taxation revenue *Rev*, positively depends upon capital and upon transfer *Tr* from federal government, which, in turn, increases when revenues are too low (in the simplest version it is constant). Current investment volume *I*(*x*,*y*) positively depends upon property tax relief *x* and profit tax relief *y*. The tax relives, in turn, positively depends upon good budget position described by *Rev*, since it provides “slack”.

All systems with positive feedback circles are apt to amplifying initial disturbances. The question is, whether the negative feedback is sufficiently strong to hamper this possibility.

In our case the most strong positive circle is: tax relieves -> more investment -> more capital -> more revenue -> more possibility to grant tax relieves. Additional one working in the same direction: more revenue -> more public goods -> better investment opportunities -> more investment -> more capital -> more revenue. These positive circles could drive two initially-different regions infinitely far from one another, if not for hampering negative circles. Capital depreciation imposes limits on the growth of K and G, exhausting of resources that can be used by capital (including labor, not presented explicitly) imposes limits on investment and capital also. Federal transfers, being dependent upon region's financial weakness, works in the same direction.

However, in the present study we have investigated only the amplifying contour connected with budget constraint. We mentioned other dependencies to sketch the context of ideas and general program of study. We see that generally prediction of divergence or convergence of different regions crucially depends upon the exact character and strength of the dependencies described. So the theoretical goal is to build logically reasonable dependencies connecting taxes with exogenous parameters and other variables. And the goal of empirical estimation is to provide reasonable exogenous parameters, under which we could explore the solutions to the model. Alternatively, theoretical results can be in the form of revealing *threshold values* of these parameters, sufficient to yield *divergence* and efficiency.

This program was reasonably fulfilled only for one-tax version of our model. We start with describing it, then we touch the version with two taxes.

2.2. Version of tax competition model with one tax (property tax). Analytical results

In this subsection we shall consider a *special case* of our 2-tax model, with several additional assumptions.

We shall suppose that only the property tax can be optimized, while other taxes are fixed. We shall suppose specific values of some constants also: $p=C^I=C^K=0$, $X^{min}=X^{fed}=0.02$, $Y^{min}=Y^{fed}=0.35$, $d=0$, $K^w=0$.

These simplifications are not too realistic, but help in analytic results.[16]

Accordingly, our **optimization problem** of r-th region becomes simpler now, with fixed $y= Y= const > Y^{fed}$:

$$A_r x_r^{(2)}+B_r x_r \;\text{–>}\; \max_x \;,\ \text{subject to}\ \mathbf{a}_r x_r^{(2)}+\mathbf{b}_r x_r+\mathbf{h}_r \geq G_r\,, X^{min} \leq x_r \leq X^{max},$$

while for the case when the constraints are incompatible due to large G_r, this optimization problem is assumed to turn into just minimizing debt [$-\mathbf{a}_r x_r^{(2)} - \mathbf{b}_r x_r - \mathbf{h}_r + G_r$] (see more comments below). Here $r<90$, G, as previously, means necessary spendings, constants A, B, H, and short-term constants **a**, **b**, **h**, were expressed in initial

[16] We shall mention even more special case with profit tax y=0, that one can interpret in the following way: profit tax and other profit-like taxes to be included into the investment curve, thus the "profit" of the curve will mean not gross profit in this section, but rather "gross profit minus profit-tax". Then investment triangle diminishes. This simplification biases the tax incidence, of cause, but allow for direct model solutions.

terms earlier (1-3) –(1-5). With new simplifying assumptions they become (we drop index r when we not need it):

$\mathbf{a}$= $(g*K^s* (y - Y^{fed}))/2 >0$; $\mathbf{b} = K^s* (1 - y + Y^{fed}) >0$;

$\mathbf{h} = K^s* (-2*g*X^{fed} + (y - Y^{fed})(1- g^{(2)}*\pi^{(2)}))/(2*g)$;

so the parabola

$\mathbf{a}\ x^{(2)}+\mathbf{b}\ x+ \mathbf{h} = K^s* (x - X^{fed}) + (K^s* (1/g + \pi - x)*(1 - g*\pi - g*x)*(y - Y^{fed}))/2$ increases on R_+ infinitely, it has minimum at some $x<0$. Hence, when budget constraint is binding, it relates to positive-sign root of the equation $\mathbf{a}_r x_r^{(2)}+\mathbf{b}_r x_r+ \mathbf{h}_r =G_r$, that is

$x^s(\mathrm{K}^s,\mathrm{pi}) = 0.5(-\mathbf{b} + \mathrm{Re}[(\mathbf{b}^{(2)} - 4\ \mathbf{a}\ (\mathbf{h} - G))^{(}0.5)])/\ \mathbf{a}$, where Re[.] denotes real part of the number. Obviously, function $x^s(.,.)$ depending upon (K^s, pi) is continuous.

Further, to understand type of solutions, note that

$A= 3\ K^s*w_1* (g*(I^m*v_3*(-2/3 + y - Y^{fed})/ K^s*w_1 + (y - Y^{fed})))/2$;

$B= (2*K^s*w_1*(y -1)*(1 - y + Y^{fed}) + I^m*(-2*g*v_4*(y -1) +$
$+v_3* (-2 - 0.04*g + 2*g*\pi + 7*y + 0.04*g*y - 4*g*\pi*y - 5*y^{(2)} - 5*Y^{fed} +$
$+ 4*g*\pi*Y^{fed} + 5*y*Y^{fed})))/(2*(y -1))$.

We shall assume further, that the first parabola has maximum, while the second one increases for all $x>0$, that can be expressed as:

1) $A_r<0$, 2) $\mathbf{a}_r > 0$, $\mathbf{b}> 0$.

It is important to note here, that there is a realistic domain of parameters for which the three properties hold. Condition "2)" is weaker then [$0< y-Y^{fed}$], that is quite realistic. Conditions "1)" under assumed positive K^s*w_1, g, I^m, becomes

$I^m*v_3/ K^s*w_1= I^{max}*C_{coeff} / K^s > (y - Y^{fed})/(Y^{fed} + 2/3 -y)$.

When $Y^{fed}=0.35$, $0.35<y<60$, then right-hand side of this relation is between, roughly, 0 and 1, while left-hand side for reasonable relation investment/capital like $I^{max}= K^s/10$, and reasonable factor $C_{coeff} =10$, is close to 1.

Thus our optimization problem becomes a convex optimization problem:

$A_r x_r^{(2)}+B_r x_r \rightarrow \max_x$, subject to $x_r \le x^s_r(\mathrm{K}^s{}_r,\mathrm{pi})$, $X^{min} \le x_r \le X^{max}$,

The above restrictions on parameters are maintained throughout this one-tax subsection. In particular, they guarantee continuity of solution to optimization problem.

Maximizing our criterion function (in a given period t) w.r.t. x_{rt} yields either the short-term-optimal solution $x_{rt}{}^s(\mathrm{K}^s{}_r,\mathrm{pi})$, when budget constraint is binding, or the unconstrained long-term-optimal solution $x_{rt}{}^*$, that will be formulated as:

$x_{rt}{}^*(\pi_{t-1}, \mathrm{K}_{r,t-1}) = \max\ \{X^{min}, \min\{X^{max}, - 0.5\ B_{rt-1} / A_{rt-1}\ \}$.

Constrained maximum $\hat{x}_{\mathrm{rt}}(\pi_{t-1}, \mathrm{K}_{r,t-1})$ relates to constrained long-term Laffer curve solution. This optimal tax choice within budget constraint, may be explained as follows. Exogenous parameters given, for some regions the critical point $x^s{}_{rt}$ that cover necessary expenditures may happen to be less than $x^*{}_{rt}$ — they will choose an optimum long-term rate of tax x^*_{rt}. The regions with a critical point between x^* and X^{max} will choose this second-best point x^s_{rt}, and other will be compelled to leave the tax rate on the upper legal bound X^{max}, not having an opportunity to reduce the current incomes for the sake of the future.[17] So, summarizing assumptions on behavior of regions, we can formulate the following continuous, piecewise- smooth function describing *response* of a given region to the changing factors in year *t*:

$$\hat{x}_{rt}(\pi_{t-1}, K_{r,t-1}) := \max\{X^{min},\ x^s_r(\pi_{t-1}, K_{r,t-1}),\ x_{rt}^*(\pi_{t-1}, K_{r,t-1})\} =$$
$$= \max\{X^{min},\ x^s_r(\pi_{t-1}, K_{r,t-1}),\ \min\{X^{max},\ -0.5\, B(\pi_{t-1}, K_{r,t-1}) / A(\pi_{t-1}, K_{r,t-1})\}\} \quad \textbf{(1)}$$

As soon as functions $x^s_r(\pi_{t-1}, K_{r,t-1})$, $B(\pi_{t-1}, K_{r,t-1})$, $A(\pi_{t-1}, K_{r,t-1})<0$ are continuous, this response function is continuos too on the admissible domain (it can be revealed substituting the expressions for $A(\ldots)$, $B(\ldots)$ into formula (1)). Obviously, the solution $\hat{x}_{rt}$ always lies within constraints $[X^{min}, X^{max}]$, but it may be not the case for budget constraint. In this case we shall also suppose formula (1) to reflect the reply-function of the region, instead of saying that there is no reply (no admissible plan). It reflects the hypothesis, that the region will do something in any case, even if it can not cover necessary expenses; either it borrows, or reduces its needs. The formula relates to minimizing the debt in this case.

Time recursion.

Competition starts with taxes $x^s_r = X^{max}$, and some initial accumulated capital K^s_{r0}. The depreciation coefficient being $\alpha<1$, new capital will be

$$K_{rt}=K_{r,t-1}*\alpha+I_{rt}.$$

In all periods besides the first, we shall assume that the regions choose the tax rate in a Nash-type fashion, expecting preservation of existing rate of net profit. They ignore strategic consequences of their decision, influencing the whole country.

Afterwards, capital market works, where capital supply is supposed linear, that is $I_0(\pi_t) = \hat{I}_0 - \beta\pi$, and equilibrium profit rate $\pi``_t$ clears the market of capital:

$$I_0(\pi``_t) = \sum_r I_{rt}(\pi``_t,\ x_{rt}),$$

that entails equation (3).

[17] We ignore a question of financing those, for whom the maximum current sum lies below critical point, they are supposed to beg from the federal budget or borrow somehow. Actually, for the ``necessary" state expenditures G_r we take a "critical" sum below which situation is considered dangerous by the head of the region.

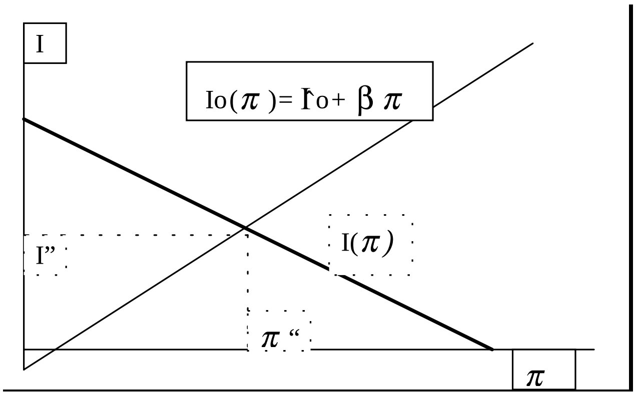

Figure 5. *A curve of investments supply Io*(π) *crossed with the total investment demand curve I* (p), *gives the equilibrium rate of net profit* π“.

Thus, we determine behavior of the game participants in each period by the following recursive system of formulae:

$x_{rt}= \hat{x}_{rt}(\pi_{t-1}, K_{r,t-1}) \quad \forall r$
(2)

$\pi_t= \pi_t(x_t) = \sum_r (\hat{I}_{rt} - g_r x_{rt} - I_0)/ (\beta+\sum_r g_r)$
(3)

$I_{rt}= I_{rt}(\pi_t, x_{rt}) = I^{max}_r*(1-g\pi_t - g x_{rt} - y+gy x_{rt})/(1-y) \quad \forall r$ **(4)**

$K_{rt}= K_{rt}(K_{r,t-1}, I_{rt}) =\alpha K_{r,t-1}+I_{rt} \quad \forall r$
(5)

$Y_{rt}=Y_{rt}(x_{rt}, K_{rt}) = x_{rt} K_{rt} \quad \forall r.$
(6)

The last equation reflects the tax revenues Y_{rt}, it is inessential for the first 5. Obviously, given the magnitudes of the last period, the system of equations (1) –(5) w.r.t. variables bundle f_t= (x_t,π_t,I_t,K_t) can be solved consistently, in the sequence from (1) to (5), and the solution is unique. So, we shall view this system as a vector function $F:R^l \mapsto R^l$, with $l:=3m+1$ arguments, defined as:

$$F(\pi,x,I,K):= (\pi(x),\ \hat{x}(\pi,K),\ I(\pi,x),\ K(K_{r,t-1},I))$$

Definition. An infinite sequence bundle $(\pi_t,\ x_{rt},I_{rt},K_{rt\, r\le m})_{t=1}^{\infty}$ satisfying the equations (1) – (5) will be named a *path* of the described economy. A bundle f” = (π”, x”, I”, K”) = ($\pi”_t$, $x”_{rt}$, $I”_{rt}$, $K”_{rt},\ _{r\le m}$) , for some moment *t*, will be named a stationary point (*equilibrium*), if it is mapped by system $F(\cdot)$ of formulas (1) –(5) into itself, so that $F(f") = f"$

By definition, the equilibrium $f"$ should satisfy the following system of equations:

$$x"_r = \hat{x}_r(I", K"), \qquad \forall r \tag{7}$$

$$\pi" = \sum_r (\hat{I}_r - g_r x"_r - I_0)/(\beta + \sum_r g_r) \tag{8}$$

$$I"_r = I^{max}_r * (1 - g\pi" - g x"_r - y + gy x"_r)/(1-y) \qquad \forall r \tag{9}$$

$$K"_{rt} = \alpha K"_r + I"_{rt} \qquad \forall r \tag{10}$$

Under our assumptions (that seem "realistic") we have existence theorem.

Theorem 1. There exists a stationary point (equilibrium) of the above model with one tax.[18]

Proof. We rely on continuity on the reply function $\hat{x}$, explained above. Continuity of other functions forming recursive single-valued function $F(.)$ can be seen in (7) –(10). So, to apply Kakutani fixed-point theorem to $F(\cdot)$ we need only compactness and convexity of its domain. According to (1), (2) the domain and the range of values for x is the m-dimensional closed cube $[X^{min}, X^{max}]^m$. Accordingly (see (3)) the domain and the range of values for π is this cube's image $\pi([X^{min}, X^{max}]^m)$. By linearity of the function $\pi(.)$, it is also a closed cube. Similarly, by definition, the domain and the range of each I_r, is also closed interval $[0, I^{max}_r]$, for all r. Let us take upper bound for capital variables $K_1,\dots,K_m$ their values defined by formula (5) or (10) with maximal possible investment: $K_r = I^{max}_r/(1-a)$, then domain for capital is $K_r \in [0, I^{max}_r/(1-a)]$. It is obvious that capital for all admissible x, I, π can not take values out of this cube, if the initial capital was inside the cube.

So, $F_t(\cdot)$ displays a convex compact domain into itself, $F_t(\cdot)$ is a continuous function, therefore Kakutani's theorem is applicable, that suffices for the proof.

Consider the case, when tax competition is going *one-way*, i.e. regions are allowed only to reduce taxes starting from some common level X^{max} (to grant tax relieves), but not to increase taxes. If in addition, somebody use this opportunity, then, obviously, there will be more investment. If somebody does not use it, he is worse off. More accurately, consider

Theorem 2. (*one-way competition*)[19] Assume that single-tax economy (1) –(5) has an equilibrium (stationary point) $f" = (\pi", x", I", K")$ where at least one region i does

[18] There can be *multiple* equilibria, as shown by examples.

use its right to grant tax relieves, that is $\exists i$: $x''_i < X^{max}$. Then this equilibrium differs from the unique equilibrium $f' = (\pi', x', I', K')$ that could be achieved without tax competition (assumption $x_r \equiv X^{max}, \forall r$) in the following features:

(i: *development efficiency*) At point f'' a larger equilibrium profit level is achieved: $\pi'' > \pi'$, and a larger total investment level I'' in the country, then the level I' achieved without competition.

(ii: *divergence*) If at point f'' there is a region j, not using its right for tax relieves: $\exists j$: $x''_j = X^{max}$, then development asymmetry of regions is stronger under competition, in the sense that each region j without tax relieves has less capital, than without competition ($K''_j < K'_j$), while at least one region using tax relieves have more investment and capital: $\exists i$: ($x''_i < X^{max}$, $K''_i > K'_i$).

The proof.

Item (i). We notice, that when tax rates $x_r = X^{max}$ are fixed, then the (equilibrium) solution to the system (7) - (12) is unique w.r.t. other variables, since the system is linear. On the contrary, the system has, generally speaking, several solutions when the tax rates x_r are free to go down. Assumptions given, some of these equilibrium values $\hat{x}_r$ ($\exists r$) will appear to be below, than X^{max}. By substituting the sum $\sum_i \hat{x}_i$ of these magnitudes into the equality (8), we have the greater profit magnitude $\pi(\hat{x}) > \pi(X^{max})$ than for fixed tax rates X^{max}. The larger profit means the larger investments according to (11), that suffices for (i).

Item (ii). Rely on larger profit rate $\pi(\hat{x}) > \pi(X^{max})$, use the assumption about existence of "critically poor region" r such, that its tax is $\hat{x}_r = X^{max}$, then, according to equation (9), this region is worse off at point f'' in comparison with equilibrium without competition, where rate of profit was less. On the other hand, the sum of the investments has increased ($I'' > I'$), hence there is at least one region winning capital, that proves (ii).

Equilibria multiplicity and different patterns of divergence and traps. Paths of the described model have been simulated on the computer, with the program in Mathematica-3.0 language, for numerous examples (see section 2.3. for algorithm and Appendix for results).

Simulations have shown, that there are multiple possible equilibria under the same exogenous conditions: fixed capital curve parameters g,q and budgets G_r (actually we normalized starting capital and G_r of regions dividing by G_r, normalization does not affect the optimization), weights w_1,w_2,δ of different goals. Varying only initial capital $K1,...,Km$ we can see finitely many different equilibria. Moreover, quite similar regions (with the same technology and goals) are winning or losing competition, dependent on initial capital (*low-budget divergence*). Alternatively, similar regions may become

[19] This one-way theorem can be reversed. Suppose the opposite: that competition starts from the lowest possible level *Xmin*, and regions have right to only *increase* taxes. Then development inefficiency will take place, by the same logic as efficiency in first variant. Besides, divergence can also take place, that depends upon the role of budget constraint. We do not touch this more, as this direction of competition is not observed in Russia.

winners or losers dependent on the weight of investments in their criterion function (*impatience-driven divergence*). These two are the most interesting patterns of divergence.

In explored examples, there was always a *bifurcation point* K_0 in terms of starting capital, such that all regions starting above this point converges into one "winners" group (asymptoteof paths), while all "critically poor" regions (whose budget constraint does not allow them to reduce taxes) – converges into separate group of "losers" (see Appendix). Divergence was absent, naturally, in those tests, where all tested regions happened to be initially above bifurcation point K_0 (all are rich enough to start competition), or below it (all are critically poor). Thus, equilibrium pattern is predetermined not only by technology and goals, but by initial state also, and low initial capital may become a "trap".

We went half-way to identify conditions, providing traps existence *necessarily*.

To do this, let us express volume of capital, sufficient to cover budget needs G, if setting maximal tax $x=X^{max}$. For this we install $x=X^{max}$ into budget constraint

$\mathbf{a}_r x_r^{(2)}+\mathbf{b}_r x_r+\mathbf{h}_r=G_r$. Substituting here expressions for $\mathbf{a}_r$, $\mathbf{b}_r$, $\mathbf{h}_r$ and solving for K^s we obtain the needed critical value of capital:

$$K^s_{\text{(critic)}}(G,g,pi,y) = (2.*g*G)/(y - 1.*Y^{fed} + g*(-2.*X^{fed} + g*X^{max(2)}*(y - 1.*Y^{fed}) + g*\pi^{(2)}*(-1.*y + 1.*Y^{fed}) + X^{max}*(2. - 2.*y + 2.*Y^{fed}))) =$$
$$= (2.*g*G)/(-0.35 + g*(0.068 + g*(-0.00056 + pi^{(2)}*(0.35 - 1.*y) + 0.0016*y) - 0.08*y) + y) =$$
$$= 22.19\ G.$$

Here we have installed the supposed in this section values $X^{min}=X^{fed}=0.02$, $Y^{min}=Y^{fed}=0.35$, and on final evaluation installed sample "realistic" values $\pi = 0.05$; $g=3.$, $y = 0.55$, to realize the order of numbers. It can be seen, that for realistic values $K^s_{\text{(critic)}}>0$. It is the main thing for bifurcation point to exist.

Like above, we denote by $f' = (\pi', x', I', K')$ the equilibrium without competition f', in formulating the below sketch of proposition (or remark). Let us denote also $f^h = (\pi^h, x^h, I^h, K^h)$ the specific "high-capital" equilibrium (attractor) that would occur if abolishing budget constraints, that is setting $G_r=0$ for all r.

Remark 1 (*Hypothetical*: *existence of low-budget trap*) Consider an economy (1) –(5) with two regions[20] with all parameters of the regions identical and fixed, but for initial capital variables $(K_1,K_2)\in R^2_+$ that can vary. Assume such parameters of the regions that their long-term maximum is below upper bound:

$x_r^*(\pi', K_{r,t-1}) = \max X^{min}, \ -0.5\, B_{rt-1}/A_{rt-1} < X^{max}$,

and critical-capital point lies between low asymptoteand high attractor:

$$K'_r < K^s_{\text{(critic)}}(G,g,\pi',y) < K^h{}_r., \ r=1,2.$$

Then critical-capital point $K_0 = K^s_{\text{(critic)}}(G,g,\pi',y)$ is a *bifurcation point*, i.e. all starting capital couples (K_1,K_2) such that $[K_1 < K_0 < K_2]$ - provides different final (equilibrium) states of capital $K_1" < K_2"$ of these identical regions 1,2.

[20] We are sure in possibility of expanding this assertion to several regions case, but its formulation becomes unnecessary tedious, since the position of bifurcation point depends upon *all* parameters and initial states of all regions.

We have not proved this hypothetical remark yet, it may happen that additional assumptions will be needed.

Fiscal efficiency of competition in general. Now consider the theoretically possible (not Russian) case, when tax competition can go *two-way*, i.e. regions are allowed both to reduce and to increase taxes $x\in [X^{min},X^{max}]$, (theorem is valid for any $[X^{min},X^{max}]$, including $[0,\infty]$) starting from some common level $x_r= X^{start}$, for all r. Then we shall see "administrative" or fiscal efficiency of competition. This effect is very general, not depending upon special assumptions, that seems strange at first glance. However, in essence it is very natural, when we compare this taxation game with monopolistic price discrimination, or with classical market, as we do in the below sketch of the proof of the below theorem, that is not proved in detail yet.

Theorem 3.(*Hypothetic . Efficiency of competition.*)

Suppose an equilibrium f" with free competition, where all regions have their "budget constraints" not binding (revenues exceed G_r for all r). Then:

i) Equilibrium f" is Pareto-efficient in terms of objectives of m regional administrations, i.e. there is no other f' satisfying demand and supply constraints (8) –(10) that Pareto dominates f".(*administrative efficiency*)[21]

ii) Assume that each region has only (long- and/or short-term) revenue objective (that is w_2=0). Then equilibrium f"gives not less total tax revenue then any equilibrium f' with uniform taxing all regions (for *any* uniform tax rate X^{start}), and this inequality is strict (">") if at least one region r has voluntarily chosen non-uniform tax $x_r\neq X^{start}$ (*fiscal efficiency*).

iii) Assume again that each region has only (long- and/or short-term) revenue objective (that is w_2=0). Compare equilibrium f" with equilibrium f' with uniform taxing all regions, having *fiscal- optimal* uniform tax rate X^{start}, that is one giving maximal total tax revenue. If equilibrium f"does not coincide with f' in tax rates, and all capital markets are open in f' (i.e. $I_1>0,I_2>0,\ldots,I_m>0$) then f" yields larger welfare loss then f', in terms of consumer and producer surplus, and gains of the state. (*welfare inefficiency, compared to optimal uniform tax, competition irrelevant for development efficiency*).

Sketch of the proof.

i) Indeed, we can look at taxation game as on buying some commodity – capital from a producer described by capital supply curve. Regions buy capital (paying by obligations of future return) and resell it to their local markets of capital (described by capital demand curves). In reality, of course, they only impose trade margins (here - taxes), but it makes no difference for the equilibrium. Regional authorities in relation to their local demands are monopolists, but in relation to producer they are like consumers, whose utility of two commodities: capital and obligations - depends upon their local markets for capital,

[21] This (i) assertion, though placed within linear-demand one-tax context, in principal is much more general, most of assumption maintained in this subsection are not essential for it as one can see from the scetch of the proof.

where they hold monopoly. These utilities may be expressed in money, or not, it does not matter. They may include tax revenue and development goals. The only thing that matter for applying 1-st Welfare Theorem to this general equilibrium on perfect market (between the producer and these pseudo-consumers) is that the objective functions should be unsatiable. They are. So, we have Welfare Theorem's assertion: any equilibrium is Pareto-efficient in terms of consumers (here - regional authorities) objectives, i.e. *administrative efficiency*. If these objectives consist only in tax revenues, then we have *fiscal efficiency* as such, otherwise we have more general pleasant feature. Important in this reasoning is that "budget constraint" G, should play no role in this trade, having another budget constraint. Otherwise it is not a classical market, and no Welfare Theorems can be applied.

ii) Exploiting our capital market linearity, we can use two ways of reasoning. One way is as follows. Investment volume I_r of a region is the demand for capital volume, linearly dependent upon its price π and upon tax x. If setting zero values for simplicity to most unessential constants, it will be: $\mathbf{I}(\pi ,x) = I^{max} (1- g\,\pi - g\,x)$. Meanwhile optimal tax rate $x^{*}= - 0.5\, B/A = S - T\,\pi$ is also linear in π, where S,T are some constants, revealed from the huge formulae of optimization section. Then in general the r-th regional demand for capital, biased by optimal taxation, behaves like $\mathbf{I}(\pi ,x) = I^{max} (1- g\,\pi - g\,(S - T\,\pi))$, that is also linear in price π.

We know from classical microeconomics that for linear market representative consumer can be found, whose utility function generates this demand. Moreover, this pseudo-consumer has quasi-linear utility function, that allows finding Pareto-optima just through maximizing sum of their utilities. Then, by (i) assertion, general equilibrium f" maximizes sum of tax revenues w.r.t. $x_1,x_2,\dots x_m$.

On the other hand, maximizing sum of tax revenues by imaginary decision-maker, choosing *uniform* tax rate x for all regions can not give more then the same task without uniformity constraint. It gives strictly less, when the results do not coincide, since the quadratic objective function is strictly concave.

Another way of proving the same, and item (iii) together, is to apply

Robinson-Schmalenzi theorem about monopolistic discrimination. Indeed, when maximizes sum of tax revenues w.r.t. $x_1,x_2,\dots x_m$, we solve, in essence, "submarket discrimination" monopolist's problem. If not coinciding with uniform-price solution, it gives strictly more profit to monopolist (that is more tax revenue in our context). The only thing, that we should realize for the proof is that maximizing their profits (tax revenues) individually, profit rate given (it is analogue of costs for them), the regions will choose exactly the same tax levels as the imaginary common decision-maker. For the case of flat capital-supply curve (analogue of constant marginal costs) it is obvious. But it is easy to see, that is the case also for all non-decreasing supply curves. This proves (ii).

Consider (iii). By Robinson-Schmalenzi theorem the total equilibrium demand on all markets is the same under discrimination and without it (if no one market closes), and welfare loss is larger with discrimination, if they do not coincide. Q.E.D.

2.2.2. *Tax competition without traps and divergence*: *discrimination between old and new capital, or no budget constraint*

We have described above the effect of divergence due to low-budget trap occurring for some region(s). However, *divergence may be absent*, when regions are able to discriminate between old and new capital, say granting holidays to investment, and fully taxing existing business (that is common in Russia now).

Remark 2. (*no traps when there are holidays*) Suppose no constraints on variables (s,t), that can be less or more then (1,1), i.e. tax holidays are available. Then there is no low-budget-traps and no related divergence effect: technologically identical regions with identical objectives always have the same stationary investment levels I", independently of initial state of capital.

Proof. Indeed, we have shown in optimization section that the 4-tax optimization problem can be split into 2 separate problems when there is no constraint (s,t) = (1,1), i.e. tax holidays are available. Then such region may grant optimal tax relieves or tax holidays to investors, paying no current price for this future benefit, that is ignoring budget constraint. So, it chooses long-term optimal strategy (s,t), after choosing some (x,y). According to the linear equilibrium relations (7) –(10), this result in similar I" for all technologically-similar regions. So, no low-budget-born divergence and no tarps occur.

Naturally, the same no-trap effect occurs, when we just do not include budget constraint into the model.

However, touching the relation of our models to reality, we can note that including into them such details as developing *public goods*, can well born traps, like budget constraint do (see the "amplifying contour" explanation, attached to Fig.3.).

2.3. Competition model with two taxes, computer simulations

Now consider more general model, then the one described in previous subsection, model with 2 taxes: property tax and profit tax (the latter was just fixed in numerical tests related to previous subsection, so this general model covers 1-tax and 2-tax cases, with or without tax holidays). It was designed for numerical and analytical study of qualitative features of tax competition process, and realized as a Mathematica-3.0 program. Let us give more detailed description of the game sequence, i.e. the computation algorithm[22] (see also Fig.2).

There are many players: m regions r=1,...,m, choosing in each period variables x_r,y_r – property and profit tax rates; s_r,t_r – tax relieves (index of time period dropped here), taking existing alternative profit p as given, together with other parameters including K,I^m (explained below); one capital market choosing profit rate p, given the taxation strategies (x,y, s,t), and one dummy player "functional dependence", determining subsequent variables I_r ,K_r . In each period, after variables (x,y, s,t, p) are settled, we determine investment volume $I_r(x_r,y_r, s_r,t_r, p)$ in each region, subsequent capital K_r(.), dependent also on accumulated capital and on fixed depreciation rate, subsequent tax revenues

[22] Some details of this general model, like public goods and federal transfers, though implemented in the algorithm, were skipped in the present series of testing the model, so not mentioned here.

$R_r(x_r,y_r,s_r,t_r,K_r)$.

Exogenous parameter $\mathbf{G}_r$ describes “necessary” spendings. When a region have deficit, then the current objective function $U(.)$ of the region changes in the direction of increasing preference weight w_1 for current revenue, so much as to achieve balance $R_r(x_r,y_r,s_r,t_r,K_r) = \mathbf{G}_r$ (increasing impatience).

This decision sequence repeats one period after another, starting from maximal tax rates X^{max}, Y^{max} and initial capitals K_1,..., K_m. In all tests it converged to some equilibria (stationary states).

Analytical study of this model was not completed, being extremely tedious.

Theorem guaranteeing the equilibria existence seem impossible without specific assumptions on parameters. Indeed, as shown in the “optimization” section and in Appendix, the functions optimized by the regions are generally non-concave on [0,1]*[0,1]. However, inexistence of equilibria in reality is hardly possible.

We started to study divergence and its welfare consequences on this general model. However, this study is hard to implement analytically, since even the reply functions have no analytical form (only algorithmical one).

Therefore, we simulated paths of the described model with two taxes on the computer, as well as for simpler one-tax model. The results showing divergence and traps of various nature are quite the same.

Conclusion

Study of optimal taxing several capital markets with immobile capital demand by *single* legislator have shown that:

1. Tax-discrimination policy, if ideally accomplished, can abandon Lafferian tradeoff between tax revenues and economic development, it enables to combine both on maximal level, that is Pareto-efficient (though in a broader context, it could hamper effective capital flows). It should be recommended when possible, if not for corruption.
2. We studied objective function combining linearly production goal and tax-revenue goal, depending upon two taxes: profit tax and property tax. Under the most realistic values of parameters, this function is concave on the narrow “ realistic” domain, and its maximum is almost always lying on the border, giving preference to profit tax against property tax, while for arbitrary parameters both statements are false, in particular, property tax is preferred for “very high” sensitivity of business to taxes (we ignored corruption and tax evasion cases, when property tax may have additional benefits).
3. Reaction of optimal taxation pattern to the exogenous parameters is as follows. Higher preference for revenue against development (employment), as well as lower sensitivity of investment to taxes, yield, naturally, higher taxes. However, only high sensitivity combined with unnaturally high preference for revenue (impatience) provides property tax becoming prefered to profit tax.

Study of game-theoretical tax competition model has shown, under the specified assumptions, that the competition has following consequences for efficiency and regional asymmetry.

The stationary point of the economy (the equilibrium) with tax competition differs from an equilibrium without tax competition by the following features.

1. When region’s short-term “budget constraint” does not play active role (that is the case, in particular, when tax holidays are practiced), *administrative efficiency* is achieved, i.e. goals of regional administrations achieved in equilibrium can not be Pareto-improved. This implies *fiscal efficiency*, i.e. maximal total country’s tax revenue – in special case when administrative goals consist only in long-term budget revenues, and when capital demands are linear. *Development efficiency* of competition in the latter case appears if and only if initial uniform tax rate was not optimal. *Welfare efficiency* (total consumer’s and producer’s surplus, plus tax revenues), on the contrary, may be enhanced or deteriorated by competition, depending upon initial state: if initial state was optimal and all markets functioned, then competition makes losses.
2. Take the case when region’s short-term “budget constraint” does play active role only for some poor regions (suppose no tax holidays, demand linearity, and other restrictions), while tax competition goes one-way, starting from the highest admissible level of tax downwards. Then *development efficiency* of competition is present, together with *divergence* (increasing asymmetry), and with usual presence of low-budget *traps*, that is isolated low-capital equilibria for some or all regions, which can be avoided if having more initial capital.

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APPENDICES

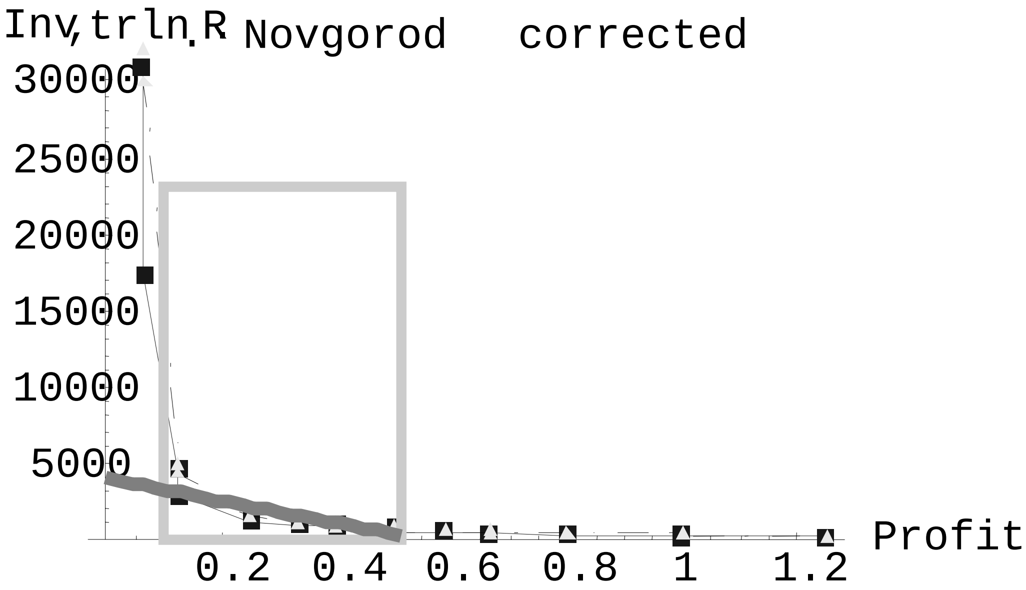
Appendix A1: Data: investment curves, their linearization

Initial data (investment projects, declared by big investors to regional administrations to get support):
Declared Investments($bln. for Novgorod, Roubles trln. for others), Declared Return-time(years) :

Novgorod: volume,	time	Buryatia volume,	time	Tyumen' volume,	time	Chita volume,	time	Hakassia volume,	time
21.9	0.08	14.988	1.9	116.3	2	128	1	8.9	1.9
1	1	19.96	2	39.3	2	483.6	1	0.27	2.5
0.5	1	37.884	2	30	2	1723.7	2	0.3922	3
0.7	1	18.538	2	20	2	820	2	0.4902	3.5
0.5	1	31.553	2	63	2.5	81.2	2.5	50.5	4
0.4	1	55.218	3	40	3.5	525	3	0.7	4
3.7	1	0.533	3	7	4	87	5	25	4
0.3	1	90	3.5	19650	5			3.216	4.5
0.5	1.25	15.019	4	3875	5				
1.05	1.5	117.42	4	8760	5				
3	1.5	4.22	4	153.7	5				
0.8	1.5	477.556	4.1	700	8				
1.4	1.7	94.772	4.5	150	10				
1.2	2	388.5	5						
0.3	2	197.166	5						
0.4	2	33.306	5						
1.8	2	19.524	5						
3.6	2	394.417	5						
0.32	2	136.625	5						
0.8	2.5	20.866	5						
10	2.5	56.406	5						
0.4	2.5	87.818	5.8						
0.3	3	171.387	6						
0.49	3	847.855	6						
19.7	4	2569.2	6.4						
112.5	8	1508.644	7						
25.3	8	207.037	8						
1000	15	173.543	8						
84	16	264	12						

Metod of building the curve and linearization (*see the programm for details*):
Announced rate of profit is calculated as 1/(time of return) and form column1 of FullTable for each region. Points (ProfitRate, InestmentVolume) describing a project from initial data are gathered in FullTable column1 and column3 respectively, descending by column1. Column2 (accumulated investments) describes sum of column3 precessing figures, each figure averaged with the next point, to fit the integral. Columns 1,2 of the resulting table with profit rate between Pmin=0.10 and Pmax=0.50 yield InvestmentTable (IT), whise data are marked by grey square on the plots. This part is linearized. The difference between initial and linearized curves above Pmin=0.10 prodices the correcting constant CI.s
----
Results:
Novgorod: PPmax= 1.25 , Kmax= 30878.8 , Imin05= 14 , Imax01= 27 =>
IT= {{1/2, 416.}, {1/2, 434.75}, {1/2, 443.5}, {1/2, 470.9999999999999}, {1/2, 538.5}, {1/2, 587.5}, {0.4, 601.5}, {0.4, 736.5000000000001}, {0.4, 866.5}, {1/3, 875.2500000000001}, {1/3,

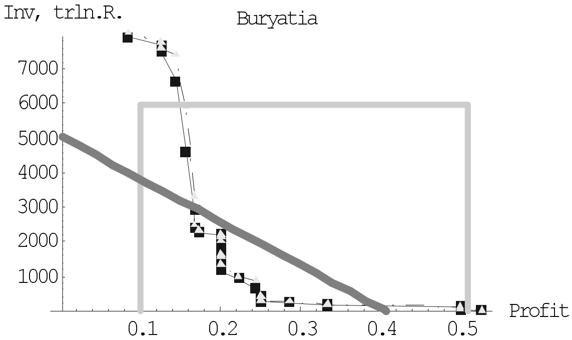
885.125}, {1/4, 1137.499999999999}, {1/8, 2790.}, {1/8, 4512.5}}



Inv(p) = 3905.74 -7338.68 p =>
Pmax= 0.532213 , g=1/Pmax= 1.87895
total profit CI0= 1124.94 , Int[Inv(p)]= 1002.65 , CI=CI0-IntInv= 122.293

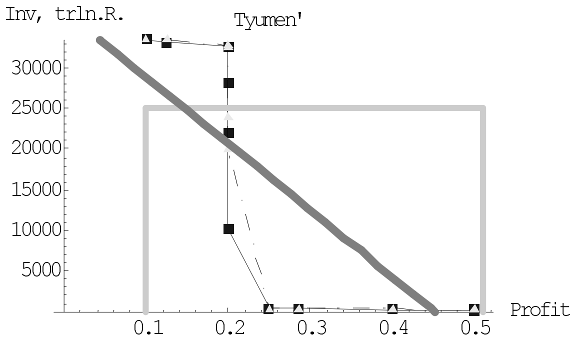
------------------------------ -------------------------- ----------------------------

Buryatia , 29,
PPmax= 0.526316
, Kmax= 7921.96
, Imin05= 2
, Imax01= 28 =>
IT= {{1/2, 24.968}, {1/2, 53.88999999999999}, {1/2,
82.10099999999998}, {1/2, 107.1464999999999}, {1/3, 150.532}, {1/3,
178.4074999999999}, {0.2857142857142856, 223.674}, {1/4, 276.1835}, {
1/4, 342.403}, {1/4, 403.2230000000001}, {0.2439024390243902,
644.1110000000001}, {0.2222222222222222, 930.2750000000001}, {1/5,
1171.911}, {1/5, 1464.744}, {1/5, 1579.98}, {1/5, 1606.395}, {1/5,
1813.3655}, {1/5, 2078.8865}, {1/5, 2157.632}, {1/5, 2196.268}, {
0.1724137931034482, 2268.38}, {1/6, 2397.9825}, {1/6, 2907.6035}, {
0.15625, 4616.131}, {1/7, 6655.053000000001}, {1/8, 7512.8935}, {1/8,
7703.1835}}



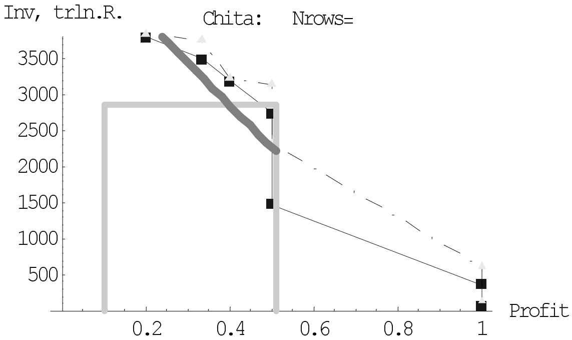
Inv(p) = 5048.16 -12421. p =>
Pmax= 0.406421 , g=1/Pmax= 2.4605
total profit CI0= 1376.76 , Int[Inv(p)]= 963.735 , CI=CI0-IntInv= 413.029

Tyumen' Nrows= , 13 ,
PPmax= 1\/2 , Kmax= 33529.3000000000063`
, Imin05= 2
, Imax01= 13



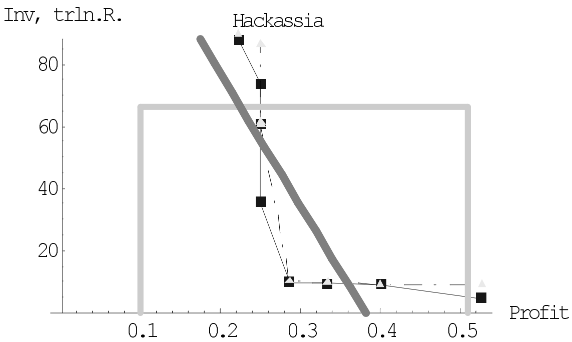
=>
Inv(p) = 37058.7 -82002.9 p =>
Pmax= 0.451919 , g=1/Pmax= 2.21279
total profit CI0= 6731.42 , Int[Inv(p)]= 7963.74 , CI=CI0-IntInv= -1232.32

Chita: Nrows= , 7 ,
PPmax= 1 , Kmax= 3805.
, Imin05= 3
, Imax01= 7 =>
IT= {{1/2, 1473.45}, {1/2, 2745.3}, {0.4, 3195.9}, {1/3, 3499.}, {1/5, 3805.}}
=>



Inv(p) = 5232.15 -5918.33 p =>
Pmax= 0.884058 , g=1/Pmax= 1.13115
total profit CI0= 2108.33 , Int[Inv(p)]= 2283.17 , CI=CI0-IntInv= -174.842

Hackassia , 8,
PPmax= 0.526316 , Kmax= 87.8604
, Imin05= 2
, Imax01= 8 =>
IT= {{0.4, 9.035}, {1/3, 9.366100000000001}, {
0.2857142857142856, 9.807299999999999}, {1/4, 35.3024}, {1/4,
60.9024}, {1/4, 73.75239999999999}, {0.2222222222222222,
87.86039999999998}}



Inv(p) = 161.31 -423.42 p =>
Pmax= 0.380969 , g=1/Pmax= 2.62488
total profit CI0= 24.8277 , Int[Inv(p)]= 28.6099 , CI=CI0-IntInv= -3.78227

The program:

```
(** Start making Inv.curve **)
ClearAll[IList0,IList1,IList,Kmax,PPmax,Pmax,Pmx,ITFull0,ITFull,ITF];
IList1={"Novgorod (1 row corrected)", 29 ,
"Investm.($bln.*25->Roubles)"," Return-time(years)",
2.19, 0.8,   1, 1,    0.5, 1,     0.7, 1,......
IList5={"Chita ", 7 ,
 "Investm.(trln.R.)","  Return-time(years)  ",
128, 1,    483.6, 1,    1723.7, 2,    820, 2,    81.2, 2.5,    525, 3,    87, 5, };

LisLis={IList1,IList2,IList3,IList4,IList5};
For[iter=1,iter<=5, iter++,
IList1=LisLis[[iter]];
 NRows=IList1[[2]]; Leng=Evaluate[2*NRows+4];
IList= IList1[[Range[5, Leng ] ]]; RegName=IList1[[1]];
(*Print["IList1=",IList1,];*)

 PiMin=0.1;PiMax=0.51;
  ITFull0 = Table[{1/IList[[2*i+2]], Sum[IList[[2*k+1]],ak,0, ia],  IList[[2*i+1]] }, ai,0, NRows-1a];
 (*=InvestFullDataTable0, we added summed investments to 3-rd column:*)
```

```
 ITFull =  Table[{1/IList[[2*i+2]],
Sum[IList[[2*k+1]],ak,0, ia]-0.5*IList[[2*i+1]],  IList[[2*i+1]] }, ai,0, NRows-1a];  (*=InvestFullDataTable, we averaged neighbore investments in 2-rd column to get closer the integral of the curve to
 the sum of 3-rd column:*)

ITF0 =Table[{ITFull0[[i,1]],ITFull0[[i,2]]}, ai,1,NRowsa];
ITF =Table[{ITFull[[i,1]],ITFull[[i,2]]}, ai,1,NRowsa]; (*=PlotListTable optimistic, while pessimistic/2 one below: *)
ITF2 =Table[{ITFull[[i,1]]/2,ITFull[[i,2]]}, ai,1,NRowsa];

For[Kmax=0;PPmax=0;Imin05=1;Imax01=1;i=1, i<= NRows, i++, Imin05=If[ITFull[[i,1]]>=PiMax,i,Imin05];
Imax01=If[ITFull[[i,1]]>=PiMin,i,Imax01];
Kmax=Max[Kmax,ITFull[[i,2]]];PPmax=Max[PPmax,ITFull[[i,1]]] ];
Imin05=Imin05+1;
IT =Table[{ITFull[[i,1]],ITFull[[i,2]]}, ai,Imin05, Imax01a];(* Selected-Data Table*) Clear[i];
Print[RegName,"
","Approximation of Inv.-Curve from FTable:"];
(*Print["IList=",InputForm[IList]]; *)
Print["ITFull=", TableForm[ITFull ] ];
Print["PPmax=",PPmax,", Kmax=",Kmax,", Imin05=",Imin05,", Imax01=",Imax01," =>"];(* bounds of Data selection*)
Print["IT=",InputForm[IT]];

 <<Graphics`MultipleListPlot`;
 LPFull= MultipleListPlot[ITF, ITF0, PlotJoined-> True ,  DisplayFunction -> Identity, (* PlotStyle -> {{Thickness[0.005], GrayLevel[0.5]}}, *)
  PlotStyle -> {GrayLevel[0.1], Dashing[{Dot, Dash}]},  SymbolShape -> {PlotSymbol[Box], PlotSymbol[Triangle]},  SymbolStyle -> {GrayLevel[0.1], GrayLevel[.9]},  PlotRange->{0,Kmax} ];

  vertices={{PiMin,-10},{PiMin,0.75*Kmax},{PiMax,0.75*Kmax},{PiMax,-10}};
Boxx=Graphics[{Thickness[.015], (*RGBColor[0,0,0.9],*) GrayLevel[0.8], Line[vertices] (*Circle[{0.7,0.7},0.7]*)}];
Inv[x_]=Fit[IT,{1,x},x];(*=Investm.Curve from approximation*)
 PIC=Plot[Inv[x], {x, 0.0, PiMax}, AxesLabel-> {"Pi","Inv"}, (*PlotRange-> {-0.1,Ymax+0.1},*)   PlotStyle -> {{Thickness[0.02], GrayLevel[0.5] (*, RGBColor[0,1,0]*)}}  , DisplayFunction->Identity ];

Show[LPFull,Boxx,PIC, PlotLabel-> RegName, AxesLabel-> {"Profit","Inv, trln.R."}, DisplayFunction -> $DisplayFunction ];

(* LP= MultipleListPlot[IT, PlotJoined-> True,  DisplayFunction -> Identity, PlotRange->{0,0.3*Kmax} ];
 Show[LP, PIC, Boxx, (*ITFull,*) PlotLabel-> "Approximated Selected Data", AxesLabel-> {"G.Profit","Investments"},
DisplayFunction -> $DisplayFunction   ]; *)
  (*Print["PiMin=",PiMin,", PiMax=",PiMax," =>"];
  Print["Imin05=",Imin05,", Imax01=",Imax01," =>"];*)
  Print["Inv(p) =", StandardForm[Inv[p]]," =>"];
 Xsol= Solve[Inv[Pmax]==0,Pmax]; Pmx=Pmax/.Xsol[[1]]; g=1/Pmx;
 Print["Pmax=",Pmx,", g=1/Pmax=", g];
 (*Print["Pmx=",Pmx,", Relative Imax= Pmx*g=1"];*)

 (**Now calculate total profit of ignored best projects, like Sum-Int:**)
 For[k=1;TotInv=0,k<=Imax01,k++, TotInv=TotInv+ITFull[[k, 3]]; CI0=Sum[ITFull[[i, 1]]*ITFull[[i, 3]], {i,1,k(*Imax01-1*)} ]  ];
Print["CI0=",CI0,", Pi>0.1-TotInv=",TotInv,"=",ITFull[[Imax01, 2]]];
 (**=Sum[Prof*Invest] of all above 10% (realised) projects **)
 IntInv=Integrate[Inv[p],{p,PiMin,Pmx}]+PiMin*Inv[PiMin];(**=Int[Prof*Invest], of all realised projects. Now the difference: **)
 CI=CI0-IntInv;
 Print["for Pi>0.10-projects total profit CI0=",CI0,", Int[Inv(p)]=",IntInv,", CI=CI0-IntInv=",CI,
 ">0.1averageProfit=",CI0/TotInv]; Print[" end ",RegName,"--------------- ---"]; ]
--------------------------------
```

APPENDIX B2. *Empirical story*: *Novgorod example*

To better realize reasons and way of thinking of an investment-seeking regional administration, we have chosen the one most known for its investment- favorable policy, that is Novgorod Velikii. We visited the city for 2 days, contacting with the administration members, with its communist political opposition, with occasional citizens. We are grateful to generous informational help of the heads of departments, especially to the head of foreign investment dept. O.Klimov, to the head of industry and communications dept. A.Mosolov. We concentrate here on motive-revealing in a journalist manner, more profound or detailed information on Novgorod investment success one can find in numerous publications, including Kuznetsova[8], [3]-Arkin et all, [21]-Polishchuk, [Y.Kantor ``Right for victory"-Izvestia 9.1999].

The facts and intuitions gathered may be summarized as follows.

1)Stylized facts on initial conditions, politics and policy.

Novgorod is not a large region (750 thousands inhabitants) with no outstanding opportunities in any sense. Its only relative advantage is being close to large consumer-goods markets like Moscow and S.-Petersburg, that is prominent mainly in comparison with Siberian regions. The main industries remains to be chemical (nitrate fertilizers and other nitrates, mainly), and electronics, depressed by low funds for weapon production. May be, it is the absence of something like oil pipe, that prevented Novgorod from too deep attention of strong Mafia groups.

In 1991 the famous now Novgorod governor M.Prusak (formerly - agricultural director) was appointed by B.Elzin, later Prusak was elected, and recently, in 1999 autumn - reelected again with wonderful for crisis-feeling Russia supporting majority (92\%). From the very beginning he tried to keep most prominent specialists in regional administration, he did not change them neither for friends and relatives, nor for businessmen. In addition, he tried to keep them out of business operations, paying enough and keeping discipline, so he has a team. He managed also to keep peace with Novgorod legislative authority, by making it a small body (28 persons) including all heads of local administrations (depending upon the governor in some aspects, as stressed by governor's opposition), and by careful considering interests and opinions (as stressed by his followers). This allowed him to quickly implement both legislative and administrative steps of his known investment-attraction policy.

2)The motives.

The reasons for which M.Prusak had started in 1992 his known effective investment-attractive policy are expressed by different referents as ``just being a decent person'' (his followers) or as ``playing a big political game, hoping for high Moscow positions'' (his opposition), that does not too much contradict on another. In any case, this governor having enough power in the region shows preference rather to maximize his glory and popularity, then current personal income welfare of his relatives (he has almost none). There were no compromise materials during his election company, and the opposition could not express clearly their impression of governor as ``dishonest'', replying to our direct questions. Moreover, the climate in regional administration seemed to be rather industrious and enthusiastic towards investment attraction and regional development, most specialists had ``team spirit'' and eagerly went into all details describing the means and tools to develop their region.

3)The means and tools.

For our study disappointing at first glance was the opinion of administration specialists, referring tax relieves as being *of secondary importance* for investment attraction, among the tools used by the governor and his team. They say, that the major factor is the confidence of investors in stability of favorable conditions, provided by administration, in administration's motivation to develop the region by good investment climate and to establish such reputation. In this context tax relieves or holidays are meaningful not as such, but rather like a *sign* of administration eagerness to play a fair game with investors. However, the specialists note, that for some firms the sum of the tax relieves amounts to considerable figures, also crucial for decision to invest in this particular region. This was the case with Cadbury-Schweps Corporation. Other tools of administration are of organizational character: - supplying possible investors with all necessary information; - attaching administration officer to every prominent project for solving problems with loaning or buying land, getting access to communications, getting all necessary permissions from medical, legal, fireman authorities, and so on.

The technology of granting tax holidays is *halfway to perfect discrimination*. The projects seeking holidays must be presented to administration, they are audited by administration and by external foreign firm with good reputation. The repay-time is detected, according to standard declared methodic, through calculation of profits. The tax holiday is always granted by administration (not by legislators) for full repay-time, that is up to restoration of capital. All presented projects get holidays, except bad in ecological aspect, or those, seeming to be a financial afera. Thus, administration though having by Novgorod law at its disposal enough power to grant relieves only to favorite enterprises, in practice treats by equal method (low or absent corruption). But it discriminates in favor of the budget, and discriminates in a fashion close to ideal discrimination theoretically prescribed: all (big) projects are treated individually (it is a big job) and stronger projects get less tax relives, proportionally. It could work even better, closer to perfect discrimination, if not for federal-law restrictions.

4)Our conclusions.

The organizational help can be considered being a cost-lowering tool,

similar to tax relief in our context. We suppose it to have the same impact, so abstein from modeling it as a different factor.

On the other hand, ``reputation'' considerations, being realistic, drive us to describe the region-investor game as a typical Bayesian equilibrium, where the investors try to realize the ``type'' of the region in question, while the region chose between ``predatory'' behavior and ``peaceful'' behavior towards investors. At the moment we are not able to enrich our Nash-equilibrium model by these considerations. However, in aggregated thinking, more or less predation looks like more or less taxes, while ``peace'' looks like tax relieves. So our arguments may well be valid for an enriched Bayesian setting also. So, summarizing, we have not thrown away our theoretical views after meeting with practical specialists.

APPENDIX C3. *Estimates of investment-curve parameters by* 5 *projects data*

For thorough estimating investment curves and idle-capital curves, one should collect comprehensive regional data on accomplished investment projects, data on idle and semi-alive facilities, data connecting profit and volume parameters of each unit of capital. We are unable to find these within the present study. Even the most informative administration – Novgorod administration – promises now to make such data available only in 2000. Instead we have at hand some data on *proposed* in 1997 large and medium-size projects, offered by entrepreneurs to 19 regional administrations and for different purposes - it is Novgorod (here projects supplied for arranging tax-relief documents) and 18 Siberian regions . These packages of projects include mostly the reconstruction and new projects reported by large enterprises. The Siberian projects within the 18 packages were written by enterprises and some new projectors for attracting various investors, including governmental ones. They were supplied to governors in reply to the request of regional authorities, some of them intended to compete for federal support, as having positive external effects on the regions. These 18 packages were ``approved'' in 1997 by governors (with unclear benefits from the approval). According to Siberian expert in investment N.Kravchenko, the volume of projects included in a package may amount to 50-70\% of all investment projects *written* in the region. In addition, demand for capital includes *unwritten* projects, implemented without external investments by enterprises, so it is more likely that a package amounts to 50\% of total region's ``investment intentions'', then hypothesis 70\%. No clear considerations were formulated about ``included'' projects being commonly more or less profitable then non-included, we shall suppose the same profitability.

Unfortunately, only 5 of the packages supply data on projects' (announced) profitability.

These profit-reporting 5 regions are:

Novgorod — typical in resources European region (but outstanding administration);
Chita — typical East-Siberian industrial and resource-extracting region (depressive),
Hackassia, Buryaia — South-Siberian resource-extracting regions
(depressive),
Tyumen' — North-Siberian oil-extracting region.

Projects at hand are described in various quality of detalization: some are just preliminary proposals, others have technical and economical calculation, or even a detailed business-plan. Each project include the needed for our purpose two figures: volume of investment and expected return-time (years). The ``return-time'', supposingly, is simply investment volume divided by annual expected gross profit. The data in tables and plots see in the end of Appendix.

Naturally, we suppose the information to be biased in several aspects.

1)Risk aspect. The announced gross profit figure implies ``everything-O'K'' assumption. So, to convert it into its "warranted" profit equivalent, needed for our model, one must diminish it several times. Relying on our intuition, we suppose, say, at least 2-3 times (so the announced net profit of Russian project should be >10\% when the investor can invest abroad for transfer-cost-cleared 5\%).

2)Intentional project author's optimism or pessimism. In those regions, where longer return-time is the legal basis for longer tax relief, the investor may have intention to overestimate return-time in his project. However, consultations with Novgorod regional authorities assured us that this tendency is neutralized by thorough economic calculations accomplished by the administration and invited auditing firms. This may not be the case in other regions. On the other hand, when a project is not launched by a sufficient-funded investor, but instead it is used by an entrepreneur to get credit, such projector has the opposite intention: to overestimate profit. This case we suppose to be more common, but how should we select ``clean'' projects? At the moment we included the correction in this respect into risk-factor estimate.

3)Low-profit projects. There are politically motivated projects proclaimed by regional administration, like electric station and international airport in Novgorod. In spite of their expected low profits, they have chance to be implemented, with the help of public funds, in some way. We should, naturally, exclude such low-profit projects from general-purpose investment curve, which describes demand for capital. Similarly, but for other reasons, we should exclude private low-profit projects, having Western-warranted equivalent of their announced profit below 5\%.

4)High-profit projects. For linear approximating investment curve, intended to reflect sensitivity of investment to taxes, we should take into account only the region presumably affected by expected shifts of tax relieves. Relying here again on our intuition, we suppose Western-warranted profit, say, between 3\% and 17\%, that is announced profit between 10\% and 50\ (not to take too narrow interval for approximation). However, high-profit projects remains in calculation of regional total taxes and tax revenues, forming the constants C^I, as explained in subsection on linearization, and in examples.

Here is an example of such correction. Initial pictures of announced profits for 5 regions see in the last Appendix, being plotted in absolute (bln. roubles) terms. The selected zone of a graph between 10\ and 50\% is approximated and the resulting function is plotted together with the dotted diagram. The diagram actually approximated does slightly differ from the initial data. It is corrected so as to eliminate systematic error distinguishing integral of the function from the square of diagram calculated as total profit of discrete projects. The results are converted into relative terms (maximal possible investments volume is taken for 1), they should be converted later into ``warranted'' or ``expected'' profit terms (dividing profits by risk-factor r, so multiplying curve slope by r). Our 5 regions showed some variety of the explored parameters:

—Investment-curve parameters revealed from announced projects of 5 regions——

```
(output of the approximating program)
```

————————————————————————————————————-

1)Novgorod, 29 projects (1 unrealistic project rejected => 28):
Threshold parameters:
PPmax= 1.25 , Kmax= 30878.8 , Imin05= 14 , Imax01= 27 = >
Results:
$Inv(p)$ =3905.74-7338.68p=>$Pmax$=0.532213,g=1/$Pmax$=1.87895 total profit C^I0= 1124.94,$Integral[Inv(p)]$=1002.65,C^{I1}=C^I0-$IntInv$=122.293, in relative terms it is C^I= 122.293/1002.65=.12197
"C^I0=44.9977*25 =1124.9 , ("Pi "TotInv) = 193.15*25 =4828.8 Roubles.
' averageProfit="0.232968
——————————-

2)Buryatia , 29 projects,
Threshold parameters:
PPmax= 0.526316 , Kmax= 7921.96 , Imin05= 2 , Imax01= 28 = >
Results:

$Inv(p)$ =5048.16-12421.p=>$Pmax$=0.406421,g=1/$Pmax$=2.4605
total profit C^I0= 1376.76,$Integral[Inv(p)]$=963.735,C^{I1}=C^I0-$IntInv$=413.029
,, in relative terms it is C^I=413.029/963.735=.42857
"C^I0="1376.76 ", ("Pi "TotInv) =7789.96 Roubles
' averageProfit="0.176736

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3)Tyumen , 13 projects,
Threshold parameters:
PPmax= 1/\/2 , Kmax= 33529.30` , Imin05= 2 , Imax01= 13 = >
Results:
$Inv(p)$ =37058.7-82002.9p=>$Pmax$=0.451919,g=1/$Pmax$=2.21279 total profit C^I0= 6731.42,$Int[Inv(p)]$=7963.74,C^{I1}=C^I0-$IntInv$=-1232.32, , in relative terms it is C^I=-1232.32/7963.74=-.15474
"C^I0="6731.42 ", ("Pi "TotInv) ="33604.3
averageProfit=0.200314

---

4)Chita: 7 projects,
Threshold parameters: PPmax= 1 , Kmax= 3805. , Imin05= 3 , Imax01= 7 = >
Results:
$Inv(p)$ =5232.15-5918.33p=>$Pmax$=0.884058,g=1/$Pmax$=1.13115 total profit C^I0=2108.33,$Int[Inv(p)]$=2283.17,C^{I1}=C^I0-$IntInv$=-174.842, C^I=-174.842 ,
in relative terms it is -174.842/2283.17=-0.076579
"C^I0="2108.33 ", ("Pi "TotInv) ="3848.5
averageProfit=0.547832

---

5)Hackassia , 8 projects,
Threshold parameters: PPmax= 0.526316 , Kmax= 87.8604 , Imin05= 2 , Imax01=8 = >
Results:
$Inv(p)$ =161.31-423.42p=>$Pmax$=0.380969,g=1/$Pmax$=2.62488 total profit C^I0= 24.8277,$Int[Inv(p)]$=28.6099,C^I=C^I0-$IntInv$=-3.78227 , in relative terms it is -3.78227/28.6099=-.1322
"C^I0="24.8277 ", ("Pi "TotInv) ="89.4684 .
' averageProfit="0.277502

Now take average figures:
$C^{Iaverage}$= (0.12197+0.42857-0.15474-0.076579-0.1322)/5=0.037404/2.
$Pmax^1$= (0.380969+0.884058+0.451919+0.406421+0.532213)/5=0.53112;
$g^{average}$= (1.87895+2.4605+2.21279+1.13115+2.62488)/5=2.0617
Average Announced Profit= (0.232968+0.176736+0.200314+0.547832+0.277502)/5=0.28707.
Solving 1-2.0617p=0, we get maximal profit p=0.48504 of the average announced curve.

****

So, our 5 regions showed some variety of the *announced* in the projects parameters:

$1.1 \le g \le 2.6$, $-0.15 * I^I \le C^I \le 0.43 \; * I^I$ (where I^I denotes the integral of the linear investment curve). To transform these parameters into *expected* parameters, we should multiply g by some risk factor. For instance, with hypothesis of risk-factor= (announced profit)/(expected profit) =2- >

"Expected" C is 0.037404/2=0.018702, "Expected" $Pmax$ is $Pmax^{average}$=0.53112/2;=0.26556, "Expected" $g^{average}$ is 2.0617*2=4.1234

Risk-factor, here taken for 2, will be better estimated in the next Appendix.

We add tables and plots of the described estimation in th last Appendix.

APPENDIX D4: *Estimates of investment parameters by statistical data and by* 19 *projects' data*.

1) Let us roughly estimate typical for Russia investment parameters *p*,*x*,*y* – alternative profit and taxes.

We suppose essential outflow of Russian capital abroad, reported by many sources. Therefore, for capital market there is an opportunity to invest abroad, say, for real warranted *net* (tax-cleared) profit rate like 5\%, i.e. p=0.05. This 0.05 will be the lower investment limit in our estimates. Further, to this we should add taxes to see expected (warranted) *gross* profit lower limit.

Normally, main 3 taxes are: 0.20 VAT, 0.02 property tax, 0.35 profit tax. It is too cumbersome to introduce VAT correctly into our model, it deserves introducing wages. For our purposes, i.e. for describing reaction of investment to tax relives, it seems sufficient to describe VAT in simplified way. That is, suppose fixed Wages/Capital ratio, then some part of VAT is attached to capital like property tax, while some is attached to profit like profit tax (that makes great difference for sensitivity to taxes). The remaining numerous taxes on business are mainly VAT-like, we shall include them similarly.

Now we should find the needed proportions. Goskomstat reported in 1996 total country's tax revenues CTR= 474541177.t.Roubles, including VAT = 134527611. (28*\%CTR*), profit tax = 100603670. = (21*\%CTR*), property tax = 35773195. = (7.5*\%CTR*), personal income tax (roughly) gives 13*\%CTR* (??). The remaining 30.5*\%* are excise (12*\%CTR*??), local taxes and other, that all will be attached to VAT in our calculations (being gathered, mostly, also proportional to return), so VAT-like taxes named VATLike = 1.0-0.21-0.075-0.13=0.585 of total tax burden of an enterprise.

We can compare this with a regional data. In country-average regional tax revenues 1996: RTR= 267718172., VAT = 42536054. =0.159*RTR*, profit tax = 64396171. =0.24RTR property tax = 35773195. =0.133RTR,

In particular, in Novosibirsk region 1998: RTR = 1838.2 b.Roubles, profit tax = 563.6 =0.306RTR , VAT = 408.1/1838.2 =0.222RTR, income tax = 251.8=0.137* 1838.2 (0.137RTR), natural resources 2.0, excise = 220.7=0.12 RTR , property tax about 0.133RTR (?), then remains 1.0-0.306-0.222-0.137-0.12-0.133=0.082 - the local taxes.

We need also profit/wages ratio. Total country's profit is 100603670./0.35 =287440000., 100603670./x=21/13, so, total personal-income-tax revenue is: x=62278000., then incomes (close to wages) are (62278000.)/0.12=518980000. Then 287440000./(5.1898×10^8) =0.55386. So, about a half of VATLike taxes can be described as profit-dependent, and half are capital- dependent. So, to express all taxes on business in 2 forms, we should increase proportionally both property tax and profit tax multiplying by (0.585+0.210+0.075)/(0.210+0.075) =3.0526.

Then average enterprise's tax structure (without tax relieves) in our model should look not like (PropertyTax,ProfitTax,Other...) = (0.02,0.35,...), but like (PropertyLikeTaxes,ProfitLikeTaxes) = (0.06,1.05) - huge result. We should take instead that VAT and other taxes double tax burden (0.04,0.70), relatively to only property and profit taxes, but not triple it. By tax relieves the first, capital- proportional component of tax burden can be reduced by 0.02, the second (profit- proportional) - by 0.22, so 0.02 <x <0.04, 0.35 <y <70 (we decrease the lower bound for profit having in mind possible tricks). We can express the same simpler: suppose roughly 1/2 of VAT and the like being related to wages, which are more or less attached to capital in most industries, and the remaining 1/2 - to profit. Suppose VAT burden being nearly the same as both other taxes together, and get (0.04,0.70). So, below warranted gross profit p=0.10 or p=0.09 it is unreasonable to invest in Russia without tax relieves or tax evasion. These figures we use as follows.

2)Let us estimate ``profit-risk-factor'' r= (Announced Profit)/(Expected Profit).

Unable to good such estimates now, let us take rough ones at least.

1-st source) Compare the revealed average planned (announced) projects profit 0.28707 with average country's profitability reported by Goskomstat: 1995: π =0.074, 1996: π =0.022, 1997: π =0.023 (biased by equity reestimation in 1996, by tax evasion, etc.).

We see announced profit being roughly 10 times more then profit of existing Russian enterprises! So, rough measure of risk factor (Planned Profit)/(Realized Profit) =10. It can be somewhat more if we take into account dead projects not shown in average statistics of profit. However, there may be as well strong factors, reducing this huge risk gap. 1)May be, the old capital is mostly obsolete, so new projects are really much better then existing facilities, at least twice, comparing $p^{\min}$ =0.05 with π =0.022. 2)Real profit for investors accumulated somewhere in offshore companies may be several times more then one reported to fiscal authorities, taking into account multiplicity of grey and black currencies, and other tools of tax evasion. For instance, aluminum industry was named in papers as hiding abroad about 5/6 of its profits, with the help of tolling. Unable to estimate such things correctly, we can only guess about risk factor being r= (Planned Profit)/(Realized Profit) <10.

2-nd source) On the other hand, for the same purpose we can examine something that we name ``volume risk-factor''. The described 18 Siberian investment-project collections for 1997 (including only *announced* big enterprise investments), compared with investments of the same regions in 1997 (including, unfortunately, all private investments, that we suppose 2 times more) shows great variety of the following kind:

2.5 < V= (Proposed Investment Volume)/(Realized Investment Volume) < 10 (average is about 4).

That is, least optimistic projectors were wrong 2.5*2 times, while best optimists overestimated their ability to find investors 10*2 times! Explaining this, we can suppose, that the gathered projects are the all available projects with *planned* net profit above 5\%. However, investors take also risk factor into account, so they invest only when *expected* profit is above 5\%. Then the optimism measure $V\in$ [5,20] can be called the volume risk-factor.

Let us try to convert these volume risk-factor figures into profit risk-factor. We can compare these [5,20] with the revealed from 5 regions average announced-investment curve $I(p)$ = $(I^m$-$gp)$ =

1-2.0617p, or more correctly, with the built before similar function, depending upon taxes also: $I(p,x,y) = (I^m\text{-}gp\text{-}gx\text{-}yI^m+gyx)/(1\text{-}y)$ =

= (1-g0.05-g0.04-0.70+g0.70*0.04)/(1-0.70)\ =1.0-0.20667g (we have substituted taxes 0.04. 0.70 and alternative profit 0.05). [Variant: smaller profit-dependent taxes 0.50 give us (1-g0.05-g0.04-0.50+g0.50*0.04)/(1-0.50)\ =1.0-0.14g]. Analogous function for expected profit is the same, but the slope parameter g must be increased r times, since profit decreases. So, we can form equation (1.0-0.20667g)/(1.0-0.20667g*r) =V for volume factor V. Substituting the revealed former factor g=2.0617 we get (1.0-0.20667*2.0617)/(1.0-0.20667*2.0617*r) =V, Solution is : r=2.3469$\times 10^{-9}$(-5.7391$\times 10^{8}$+1.0$\times 10^{9}V$)/V=2.3469(1-0.57391/V) ,

[Variant with 0.04,050: (1.0-0.14g)/(1.0-0.14g*r) =V, Solution is : r=0.14286(-50.0+7.0*2.0617+50.0V)/V = (7.143-5.0813/V) .]

Now substitute ``volume risk-factor'': 20->$r$=2.3469(1-0.57391/20) =2.2796 - maximal profit-risk factor . Analogously, minimal ``volume risk factor'' r=2.3469(1-0.57391/5) =2.0775.

These figures 2.1<r<2.3 are free from tax evasion bias, since they come from the revealed decision to invest. They may be biased by (all projects)/(big projects) <2 difference, but we see that with somewhat different V the solution show almost the same figures, it is non-sensitive to this bias.

Note, new average figure r=2.2<10 is lower then 10 revealed from Goskomstat's "profitability". So, we can guess country's average profit-risk factor (Planned Profit)/(Realized Profit) being about 2.2. Then average investment curve in expected-profit terms must look like $I(p)$ =1-2.06*2.2*p, that is moderate sensitivity . It seems acceptable. Similar calculations with greater alternative net profit 0.09 showed us lower risk-factor like r=1.4, also acceptable.

Really, too big risk-factor r and big sensitivity factor g both would seem unrealistic in common intuition. May be, if really having average 2.2% profit, Russian businessmen would be so sensitive to tax relief for 2% (like total abolishing of capital tax) as to increase investments for 0.4 units (where unit is maximal total possible investment=1), as predicted by the figure g=20 (coming from Goskomstat figure r=10). But it is hard to believe, and not reported by experts like Novgorod administration.

We should now try this logic on panel data to better reveal mean values g and r.

3-rd source) *Estimates of investment sensitivity to taxes* (*or to alternative profits*) *from regression.*

As reported in econometrics part, we have some regression estimates of investment sensitivity to ``Investment law'' adoption. The law usually starts tax relieves for investors (and start institutional benefits also, that biases our logic, but we shall ignore it). The average (across the country) sensitivity of investment to the law amounted to 0.135 or 0.20 increase in relative volume of investment (under different methods examined). We should convert this figures now into investment-curve terms.

Let us suppose, more or less typical for the ``Investment law'', 5 years of full tax relief for both regional taxes in question (0.02 property tax, 0.22 profit tax). What it means in terms of present-value-equivalent profit?

With supposed ``impatience'' discount 5%, 5-years unit of profit gives present value (utility) $\sum_{t=1}^{5} 1/1.05^t$=4.3295, while eternal profit unit brings $\sum_{t=1}^{\infty} 1/1.05^t$=20.0, that is 4-5 times more. So, with discount 5%, 5-years tax relief bringing profit like 0.02 (temporary property tax abolishing) is equivalent in present-value terms to roughly 0.005 eternal relief of this tax. Analogously, we suppose 5-years profit tax relief for 0.22 being equivalent to eternal 0.05 profit tax relief .

Let us substitute this into the investment curve and formulate the equation to find "expected'' sensitivity slope g (thereby we shall find risk factor r):

$I(p,x,y) = (I^m - gp - g(x-0.005) - (y-0.05)I^m + g(y-0.05)(x-0.005))/(1-(y-0.05))$ –>

substitute [$I^m=1, p=0.05, x=0.04, y=0.70$]:

$[(1-g*0.05-g*(0.035) -0.65+g*0.65*0.035)/(1-0.65)]/[(1-$
$- g*0.05-g*(0.04) -0.7+g*0.7*0.04)/(1-0.7)]=$
$= 2.8571(0.35-.06225g)/(1.0-0.20667g) =1.135,$

Solution is : g=2.3805 . This is the lower estimate. We get r=2.3805/2.06=1.1556.

[Variant with p=0.09- > $(1.0-0.29214g)/(1.0-0.34g)$ =1.135, Solution lower is : g=1.4398 . Another solution (upper estimate) is : g=2.1756]

Let us solve the same with upper estimate of sensitivity:

$2.8571(0.35-.06225g)/(1.0-0.20667g) =1.4$, Solution is : g=3.5881 . So, the announced mean sensitivity g=2.06 from the 5 projects is about 1.2-1.5 time lower then g here. This means risk factor r=1.15-1.7418 , somewhat lower, then the revealed from other calculations.

Thus, the estimates of revealed risk factor varies about $1.15 \le r \le 1.74$. So, average figures like r=1.5,g=3.0 revealed from the projects seem reasonable.

3)Estimates of variations across regions of investment sensitivity to taxes g.

Let us come back to volume risk-factor revealed from 18 Siberian investment-project collections for 1997. Its fluctuating from 5 to 20 may be explained as risk-factor fluctuation across regions, or in opposite direction - as announced-sensitivity factor fluctuation. Indeed, suppose the same across regions risk factor r=1.5 and solve w.r.t. g the equation for the least-optimistic region:

$(1-g*0.05-g*(0.04) -0.7+g*0.7*0.04)/(1-1.5g*0.05-1.5g*(0.04) -0.7+1.5g*0.7*0.04) = (0.3-.062g)/(0.3-0.093g)$. Equalize this to least volume risk-factor $(.3-.062g)/(.3-.093g) =5$,

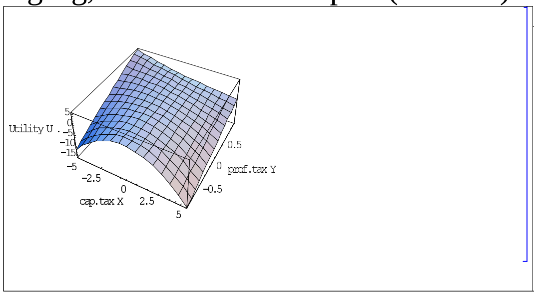
Solution is : g=2.9777 .

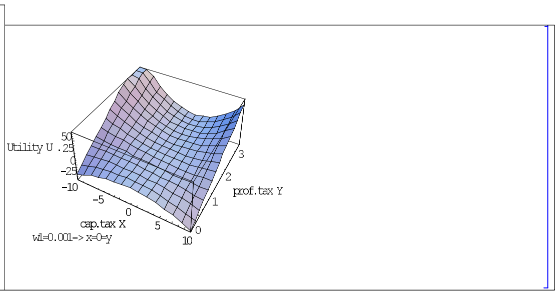
Analogue for most optimists give us (.3-.062g)/(.3-.093g) =20, Solution is : g=3.1702 . So, risk-factor like 1.5 fits well for least-sensitive-to tax regions, and something like r=1.7 - for more sensitive ones. We see also, that small sensitivity g variations may bring substantial jumps in (AnnouncedDemandForCapital)/(RealInvestment) =V ratio, and vice verse - g is revealed from figurs (5,20) with minor variations. So this method can not reveal variation of g, and in sample calculations we should take larger variations revealed from the 5 projects.

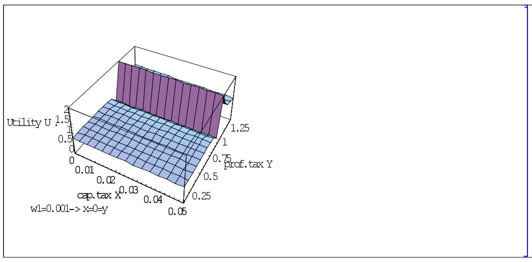
Summarizing, we should try in our calculations the ``reasonable" parameters $2.0 \leq g \leq 4.0$, X^{min}=0.02, X^{max} =0.04, Y^{max} =0.70, Y^{min}=0.40 (we have taken 0.22 for possible tax relief for profit and 0.08 for local taxes, subtracted these from 0.70), $0.13 \leq Y^{fed}$=0.26 (the doubled according to VAT-like taxes figure 0.13) , $p \geq 0.05$. Besides, "preference for current revenue" parameter w_1 above 0.3 or 0.5 would mean "extremely impatient" governor (as revealed roughly in "dynamic section"), so reasonable is w_1<0.5.

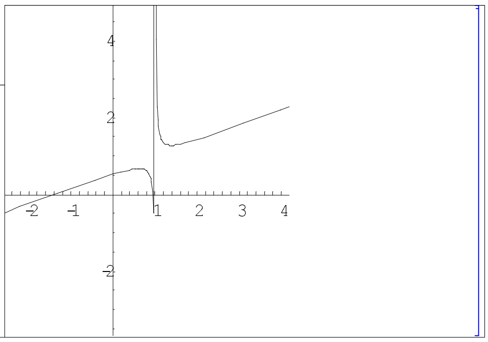
APPENDIX E5: Images of the tax-optimizing objective function: plots

Under most parameters, the function within the admissible square looks concave and pleasant; but for wider range it can be saddle-like , while on unrestricted domain it shous zigzag, well seen on 4-th plot (x=const):









>For unreasonable parameters the U surface can be wild like the saddle above. See reverse parabol for X: Taking Yfed=0.13; Xmin=-00; Xmax=1; Ymin=0; Ymax=0.99; Km=1; Imm=1; q=1; g=2; pk=0.05; p=0.0; w1=1; w2=1-w1; w3=0; w4=0; Partially-convex shape of U= ((0.95-x-y+x*y)*(x+(-0.13+y)*(1/2-x+(0.95-x-y+x*y)/(2*(1-y)))))/(1-y); at the point: U=0.3999999999999998-0.6533333333333333*x+0.08000000000000007*x^2]}

APPENDIX F6. Description of the concavity numerical test

Concavity, or at least quasi-concavity of our utility function $U(x,y)$ is needed to be sure in absence of several isolated local maxima. It helps optimization, and helps proving that optimal decisions (*Xopt*[*K,p,...*], *Yopt*[*K,p,...*]) are continuous functions of parameters, that enables proving equilibria existence.

It is easy to check numerically (see our plots, with zigzag w.r.t. y argument), that on the broad admissible square $(x,y) \in [0,1]*[0,1]$ of optimization arguments the needed properties are *absent*. However, much smaller area $[X^{min}, X^{max}]*[Y^{min}, Y^{max}]$ is practically important, and the properties may well hold on it, so we tested this hypothesis.

The function $U(x,y)$ being too cumbersome, it is hard to analytically find the exact parameters domain, providing concavity or quasi-concavity. So, we decided to test this empirically by direct multiple random test, searching for the triples of points violating concavity or quasi-concavity of the function.

The test was implemented as follows. Generally speaking, we should search across all admissible values of all function parameters:

$q,g, pi,p, w_1,w_2, v_3, v_4, y,x, X^{fed},Y^{fed},K^s,K^w, Imax, X^{min}, X^{max}, Y^{min},Y^{max}, C^I,C^K$.

Actually, we fixed some parameters on their "most probable" level (see empirical argumentation in Appendix):

pi=0, p=0.05, X^{fed} =0, Y^{fed}=0.35, X^{min}=0.02, X^{max}=0.04, Y^{min}=0.35,Y^{max}=0.60, $Imax$=1, K^s=10, K^w=1, (*i.e.* d=0.1), C^I=0, C^K=0,

while preference for development $w_2=1- w_1$ was attached to w_1, and v_3, v_4 were attached to w_1,w_2, like $(v_3, v_4) = (w_1,w_2)*Coeff$, where $Coeff \in [3,15]$ was varied with step 1: *Coeff* = 3, 4, 5, ...,15, (it describes different impatience of decision-maker). Other parameters were varied in intervals $q \in [2,4]$, $g \in [2,4]$, $w_1 \in [0,1]$.

The program (algorithm) in Mathematica-3.0 language application** included the nested cycles of varying parameters [q, g, w_1, *Coeff*], and *Niter*=500, or 5000 tests of different triples of (x,y) values for each given values of [q, g, w_1, *Coeff*].

Each tested "triple" consisted of two random points (x,y), (x',y') (two values for each of the two taxes) and third point in between, taken randomly on the interval $[(x,y), (x',y')]$. Then we compared value of the utility function in the three points, testing concavity and quasi-concavity (the middle must be not less then the minimum of the ends). If concavity or quasi-concavity were violated, cycle would be stopped and result printed.

There were two types of tests with different initial parameters:

*First***:

One cycle included *Niter*=5000 steps with capital tax X varying from 0.02 up to 0.04, profit tax Y from 0.3 to 0.6. There were 3 nested cycles: weight of revenue in the objective function (w_1) running 11 points from 0 up to 1 with step 0.1, weight of future: Coeff - from 3 to 15 (step 1) and coefficient of sensitivity of capital and investments to taxes: q,g, - going from 2 to 4 (step 0.02). Constant values are described above.

Second:

This type is differs only in 3 features: q and g coefficients of sensitivity were taken (running trough the interval from 2 to 4 with step of 0.02) independently, and one cycle included 500 (instead of 5000) steps of random "triples". Besides, Coeff was varying from 0 up to 15 with step of 1.

There were obtained negative results in both types of tests: we have not found points in the considered area where concavity or quasi-concavity was violated. Therefore, it seems plausible that the function is concave in examined interval.

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APPENDIX G7. Optimal tax mixtures under different parameters

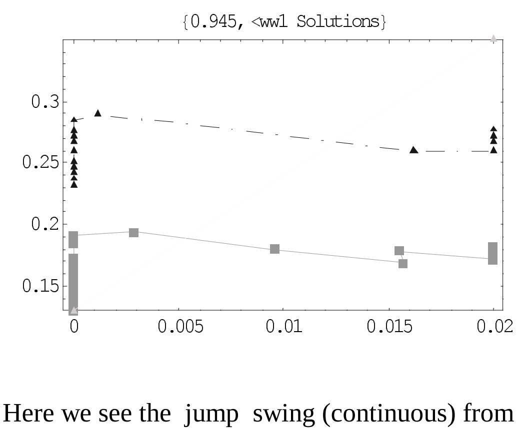
Two-tax direct iterative optimization: sample calculations

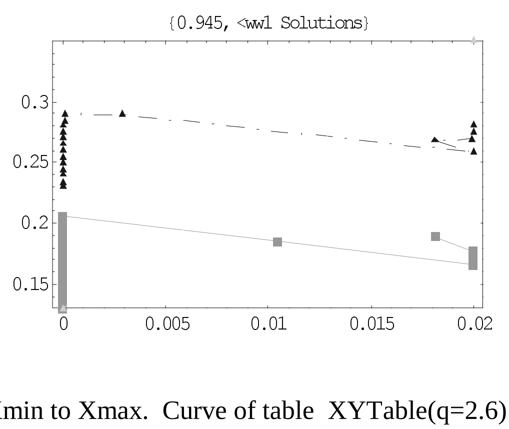
Special case of objective function was optimized: w_1=0, w_2=0, that is the same as taking K^s=0, K^w=0. When main varying parameters took very high values 3.6< g <3.9, 0.92 < w_1 <0.97 – then property tax X may be chosen not on its lower bound.
On the below plots abscissa is property tax X, ordinate is profit tax Y.
Most calculations give trivial solution: (x=0, y>0). Here we gathered only the nontrivial cases, showing the threshold values of preference for revenue W1, sufficient to both taxes play some role. This W1 appears to be too high in all examples, so for usual values (x=0, y>0) is the norm.

1) Low-taxes seria of tests:
Example of interior maximums and swing (almost jump) when 0.90<w1<97
Xmin=0.00; Xmax=0.02; Ymin=0.13; Ymax=0.35; Xfed=0.00; Yfed=0.13;
Km=1;Imm=0; q=2.6(=2.3); g=2; pk=0.18; p=0.09; w1=0.85-0.98 (step 0.005); w2=1-w1; w3=0; w4=0; s=1; t=1;

XYTabl(q=2.6) ="
{{0., 0.13}, {0., 0.1371461190211576},
{0., 0.1431747655122655}, {0., 0.1490627856878243},
{0., 0.151}, {0., 0.1554420122919736}, {0., 0.1614405512530125}, {0.,
0.1672379329004329}, {0., 0.1731835944026733}, {0.,
0.1847147044390078}, {0., 0.1873333333333333}, {0.,
0.1910483794799971}, {0.002852377394277547, 0.1939079365079365},
{
0.00954666310856326, 0.1801200980392156}, {0.01569821428571428,
0.1691666666666666}, {0.01554499999999999, 0.17825}, {0.02,
0.1717619047619047}, {0.02, 0.1751079867586833}, {0.02,
0.1813489039555216}}

XYT2(q=2.3) =" {{0., 0.2320207830825864}, {0.,
0.237210850041771}, {0., 0.242244246031746}, {0., 0.2472683608058608}, {
0., 0.2465864747180923}, {0., 0.2517535647917613}, {0., 0.26}, {0.,
0.2672555261173295}, {0., 0.2722902275270309}, {0.,
0.2718749781006747}, {0., 0.2767710901027077}, {0.,
0.2846547619047619}, {0.001112377394277546, 0.2896928860613071}, {
0.01621339285714286, 0.26}, {0.02, 0.26}, {0.02, 0.2670287698412698}, {
0.02, 0.2722253465746499}, {0.02, 0.2721399464656431}, {0.02,
0.2773766456582633}}





Here we see the jump swing (continuous) from Xmin to Xmax. Curve of table XYTable(q=2.6) is grey. Horisontal axis is Xopt, ordinate is Yopt. Nearby is the same solution with another starting approximation and more iterations: We see that greater slope of investment curve g=0.26 implies less optimal taxes then g=0.23, but anyway under sufficiently high preference for revenu both of them tennds to upper bounds. With peference below 0.85 only profit tax Y is active.

High-taxes seria of tests:

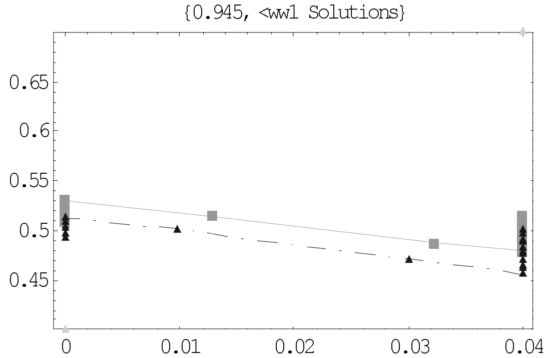
Test 2) The same test, but with higher Yfed , and (unreasonably) low Xfed (otherwise no switch to profit tax, unless too high revenue-preference W1).

Again interior maximums and switch (almost jump) when 0.90<w1<97

```
Xmin=0.00; Xmax=0.04; Ymin=0.40; Ymax=0.70; Xfed=0.005; Yfed=0.36;
Km=1;Imm=0; qq=2.3;q=qq; g=0; pkk=0.09; pk=pkk; p=0.00; (*w1=0.960;*)
w2=1-w1; w3=0; w4=0; s=1; t=1;Wstart=0.85;
```

"XYTabl(q=" 2.3
") =" {{0., 0.5080910921036476}, {0., 0.5100222688776804}, {0., 0.5151244897959184}, {0., 0.5177780612244898}, {0., 0.5211927878929759}, {0., 0.5249118772719169}, {0., 0.5303193452380952}, {0.01292775211945886, 0.5146190476190476}, {0.0322725200766638, 0.487599097763223}, {0.04, 0.4790189143073183}, {0.04, 0.481874320185699}, {0.04, 0.4863742657977232}, {0.04, 0.4901135949922422}, {0.04, 0.4942157851020133}, {0.04, 0.4983376231233775}, {0.04, 0.5024806578838397}, {0.04, 0.5066002380031101}, {0.04, 0.5107586565674311}, {0.04, 0.5148970388724074}}

"XYT2(q=" 2.4
") =" {{0., 0.4920247701992418}, {0., 0.4969285714285715}, {0., 0.5016991691408038}, {0., 0.5038579955760363}, {0., 0.5090770142202984}, {0., 0.5126639572577072}, {0.009849251700170571, 0.5009721311863656}, {0.03009917187746971, 0.4706610372340426}, {0.04, 0.4561745958444539}, {0.04, 0.4623373084546342}, {0.04, 0.4649205079675428}, {0.04, 0.4710886154296128}, {0.04, 0.4754774113574564}, {0.04, 0.4781033186819779}, {0.04, 0.4824900400755151}, {0.04, 0.4886831693365369}, {0.04, 0.4913190146442014}, {0.04, 0.4969285714285715}, {0.04, 0.5001935061345262}}



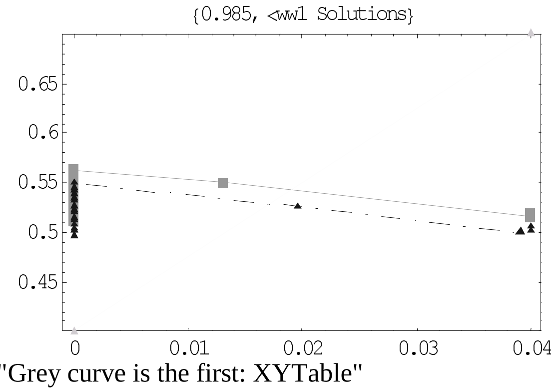
"Grey curve is the first: XYTable"
"[Km,Imm,w1,w2,w3,w4,q,g,pk,p,s,t, Xmin,Ymin]=" 1 "," 0 "," w1 "," 1-w1 "," 0 "," 0 "," 2.3 "," 0 "," 0.09 "," 0. "," 1 1 "," 0. "," 0.4 ", WW1 belongs to interval =" {0.855,0.945}
{0.855,0.945}

Test 3) The same test, but with higher Yfed and Wstart:, with too high profit rate p=0.09, with somewhat more reasonable Xfed but too high revenue-preference W1 are needed for switching taxes to profit tax X.:

```
Xmin=0.00; Xmax=0.04; Ymin=0.40; Ymax=0.70; Xfed=0.005; Yfed=0.26;
Km=1;Imm=0; qq=2.3;q=qq; g=0; pkk=0.09; pk=pkk; p=0.00; (*w1=0.960;*)
w2=1-w1; w3=0; w4=0; s=1; t=1;Wstart=0.89;
```

"XYTabl(q=" 2.3
") =" {{0., 0.511065297216694}, {0.,
0.514570591859502}, {0., 0.5162762234827509}, {0.,
0.5197792843088127}, {0., 0.5232971966624091}, {0.,
0.5268015375809423}, {0., 0.5320906681324725}, {0.,
0.5338005762873732}, {0., 0.5373276721240212}, {0.,
0.5426124986227209}, {0., 0.5461494625868049}, {0.,
0.5496720220591678}, {0., 0.5532149929298934}, {0.,
0.5567605802770597}, {0., 0.5603162434708522}, {0.,
0.5628887755102041}, {0.01308904583375438, 0.55}, {0.04,
0.5157413068007123}, {0.04, 0.5197477991196479}}

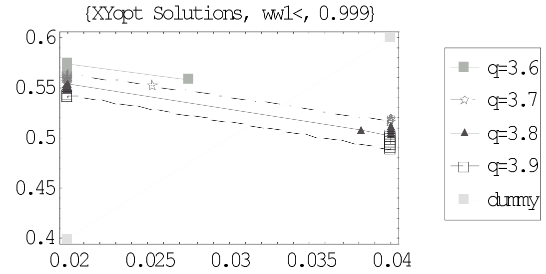
"XYT2(q=" 2.4
") =" {{0., 0.4969285714285715}, {0.,
0.5020084091909046}, {0., 0.5039350103308235}, {0.,
0.5076608681009009}, {0., 0.513186868232281}, {0.,
0.5151188656943404}, {0., 0.5206363883713}, {0., 0.5226066687226607}, {
0., 0.5263512237224981}, {0., 0.5318862676196959}, {0.,
0.5356467860226599}, {0., 0.5394159015344853}, {0.,
0.5431964858707257}, {0., 0.5452056138261957}, {0., 0.55}, {
0.01955239087471502, 0.5257630151339713}, {0.03910159028658682,
0.4999955126503571}, {0.04, 0.5018568791333363}, {0.04,
0.5055713115981726}}



"Grey curve is the first: XYTable"
"[Km,Imm,w1,w2,w3,w4,q,g,pk,p,s,t, Xmin,Ymin]=" 1
"," 0 "," w1 "," 1-w1 "," 0 "," 0 "," 2.3 "," 0 "," 0.09 "," 0. "," 1 1 "," 0. "," 0.4 ",
WW1 belongs to interval =" {0.895,0.985} {0.895,0.985}

The program used is an heuristic iterative procedure, not using derivatives, only checking function values.

Test for various slopes q of investment curve:

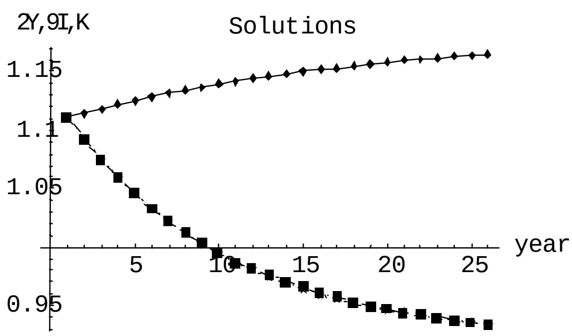


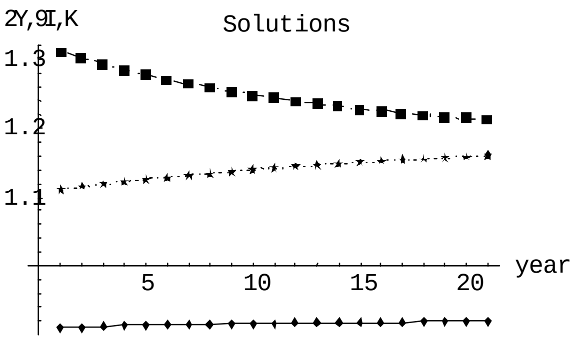
We see zigzag (up-right-up) respond of optima to "preference for revenue" *W*1 parameter, for 4 alternative demand-slope parameters *q*=3.6, 3.7, 3.8, 3.9 : larger *q*-yields lower taxes.

APENDIX H8: Divergence or convergence: the tax-competition- game model simulated by computer

On the below plots abscissa is time, ordinate is capital *K*(*t*).

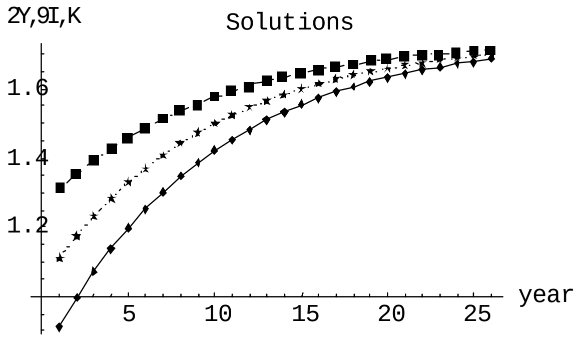
Test1: Divergence of 2 regions with different goals W1:

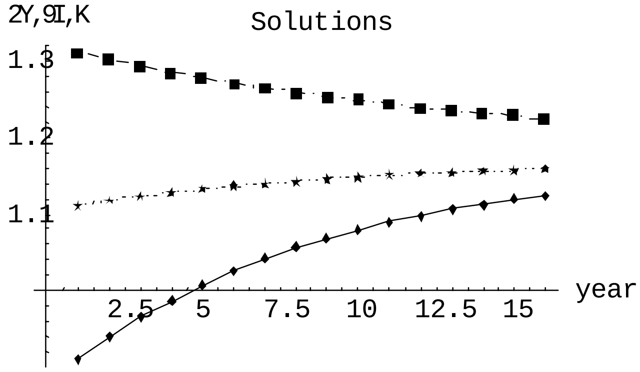




=Test2: Divergence of 3 regions with different K(r), qq=3.8; bifurcation point is between 1-st and 2-nd the regions, higher asymptoteis among the first and the second.

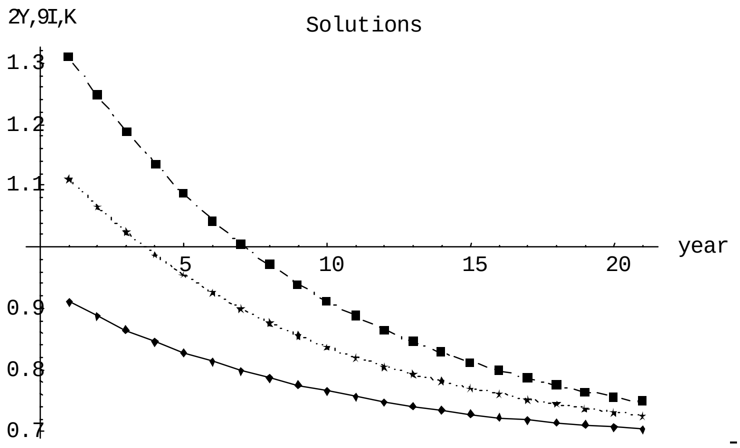
Test3: Convergence; the same 3 regions, but low sensitivity qq=0.8, high asymptote:





= Test 4: Convergence, with large slope q, too low G=0.033 gives convergence; i.e. bifurcation point is below all starting capitals, the asymptote among them:

Test N5 : Convergence: The same regions with "weak" capital market, :



: Low bifurcation point; here too high G gives convergence; i.e. bifurcation point is below all starting capitals:

** ClearAll[q,g,pi,p,w1,w2,v3,v4,y,X1,X2,Aver,x,Uaver,U1,U2,U12,Xfed,Yfed,Ks,Kw, Coeff,Uxy,Imax,Xmin,Xmax,chisloIt,Y1,Y2,Ratio,Ymin,Ymax,AverX,AverY];

```
p:=0;
pi:=0.05;
Ks:=10;
Kw:=1;
Imax:=1;
Xfed=0.02;
Yfed=0.35;
Uxy=w1*(x- Xfed)Ks+w1*(y-Yfed)Ks *0.5 (pi+x+1/g)(1-g pi-g x)+
   Kw(1- q p-q x -y+q y x)/(1-y)*
     (w1*(x- Xfed+(y-Yfed)*((0.5+0.5(1-q p-q x -y+q y x)/(1-y) –q x )/q))+
       w2)+Imax(1- g pi-g x -y+g y x)/(1-y)*
     (v3*(x- Xfed+(y-Yfed)*((0.5+0.5(1-g pi-g x -y+g y x)/(1-y) –g x )/g))+
       v4);

q=g;
w1=1-w2;
v3=1-v4;
Xmin=0.02;
Xmax=0.04;
Ymin=0.3;
Ymax=0.6;
chisloIt=5000;

                                    Print["Begin:"];


          For [w1=0,w1<=1,w1=w1+0.1,
                    Print[w1];
                    For [Coeff=3,Coeff<=15,Coeff=Coeff+1,
                              For [g=2,g<=4,g=g+0.2,

                                                  For [Nriter=1,Nriter<=chisloIt,Nriter++,


          Clear[X1,X2,AverX,x,Uaver,U1,U2,U12,Y1,Y2,AverY,y,Ratio];
                                                  X1=Random[Real,{Xmin,Xmax}];
                                                  X2=Random[Real,{Xmin,Xmax}];
                                          Y1=Random[Real,{Ymin,Ymax}];
                                          Y2=Random[Real,{Ymin,Ymax}];
                                                  Ratio=Random[];
                                         AverX=X1*Ratio+X2*(1-Ratio);
                                                  x=AverX;
                                                  AverY=Y1*Ratio+Y2*(1-Ratio);
                                          y=AverY;
                                                  Uaver=Uxy;
                                                  x=X1;
                                         y=Y1;
                                                  U1=Uxy;
                                                  x=X2;
                                         y=Y2;
                                                  U2=Uxy;
                                                  U12=U1*Ratio+U2*(1-Ratio);
                                                  If [Uaver<=U12,
                                                                              If[Uaver<=Min[U1,U2],
```

```
                                                  Print["Function Uxy is not
quasiconcave in Iteration number",Nriter,

        "Uaver,U1,U2=",Uaver,  U1,   U2];
                                                 Goto [theend];
                                                ,];
                                                Print["Function Uxy is not concave
in Iteration number",Nriter,

        "Uaver,U12=",Uaver,  U12];,,];
                                  ];];];];
Print["the END1"];
Label[theend];
Print["The END"];
```

//////////////////////////OUTPUT////////////////////////////////

```
Begin:

0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.
the END1
The END
```